Generic Reals and Proper Forcing

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"All who have meditated on the art of governing mankind have been convinced that the fate of empires depends on the education of youth."

-Aristotle

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Abstract and Acknowledgments

Absract

In this work we study the notion of forcing and show some examples of how to use it to add generic real numbers to a model of set theory. We compare the properties from these examples, and show that they do not yield the same generic extensions. We then introduce the idea of iteration of forcing notions that we will use to create a model from the examples given. Before we do, however, we investigate the notion of properness, and show that it is preserved by countable support iterations. We then prove a few results about the types of reals added by different steps of countable support iterations of proper forcing notions. We end with an example using the generic reals introduced in the beginning and the theorems of proper forcing.

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1 Preliminaries

In this chapter, we introduce some of the basic notions of set theory, and a few well known results of the real line, including a few topological properties. Any nonstandard notation will be introduced as needed, most notation will remain consistent with the books by Jech [8] and Kunen [10].

The Real Line

We assume the reader has a basic introduction to modern set theory, and understands basic notions such as the integers, $\mathbb{N}$ or ω, and the rational numbers $\mathbb{Q}$. We will show three ways one might proceed in defining the set of real numbers $\mathbb{R}$ from these basic notions. We will show in later chapters that $\mathbb{R}$ is not absolute like $\mathbb{N}$ and $\mathbb{Q}$, but can differ depending on properties of the model of set theory used.

1.1 Dedekind Cuts

Definition. A *Dedekind cut* in $\mathbb{Q}$ is a pair (A, B) of disjoint nonempty subsets of $\mathbb{Q}$ such that:

1. $A \cup B = \mathbb{Q}$
2. $a < b$ for any $a \in A$ and $b \in B$
3. A does not have a greatest element

We now say that $\mathbb{R}$ is the collection of all of the Dedekind cuts in $\mathbb{Q}$. We want of course that $\mathbb{Q} \subseteq \mathbb{R}$, and luckily for us, there is a very natural embedding. If $q \in \mathbb{Q}$ define $A_q := \{x \in \mathbb{Q} : x < q\}$, and $B_q := \{x \in \mathbb{Q} : x \geq q\}$, then the pair (A_q, B_q) is a Dedekind cut in $\mathbb{Q}$.

We equip $\mathbb{R}$ with the usual topology generated by the rational intervals, and the usual metric.

1.2 The Cantor Set

Definition. The *Cantor Set*, or the *Cantor Space* is the set $2^\omega := \{f : \text{“}f \text{ is a map from } \omega \text{ to } 2\text{”}\}$.

We equip 2^ω with the following topology. Let $p \in 2^{<\omega}$ (a binary sequence of finite length), then define the set $U_p \subseteq 2^\omega$ by $U_p := \{x \in 2^\omega : p \subseteq x\}$. Then $\{U_p : p \in 2^{<\omega}\}$ builds a base for our topology.

We can also equip the Cantor space with the following metric. If $x, y \in 2^\omega$, and $x \neq y$ define

$$\Delta(x, y) := \min\{n \in \omega : x(n) \neq y(n)\}$$

Then define the metric on 2^ω by

$$d(x, y) := \begin{cases} 2^{-\Delta(x,y)} & \text{if } x \neq y \\ 0 & \text{if } x = y \end{cases}$$

The metric topology generated by d is the same as the topology defined above. It is easy to see that this space is completely disconnected, as the base we defined above is one of clopen sets. We could "glue" this space together by the following method. Let $a := \langle 0, 1, 1, 1, 1, ...\rangle$, and $b := \langle 1, 0, 0, 0, 0, ...\rangle$, then if $x \in 2^{<\omega}$ is any finite seqence of 0's and 1's, we say $x^\frown a \sim x^\frown b$ (where $^\frown$ is concatenation). We claim now that this is an equivalence relation, and the quotient space $2^\omega / \sim$ is homeomorphic to the closed unit interval $[0, 1]$.

1.3 The Baire Space

Definition. The *Baire Space* is the set $\omega^\omega := \{f : \text{“}f \text{ is a map from } \omega \text{ to } \omega\text{”}\}$.

Similarly to the Cantor space, we equip the Baire space with the topology built by the sets $U_p := \{x \in \omega^\omega : p \subseteq x\}$ where $p \in \omega^{<\omega}$.

We can use the exact same metric as we used on the cantor space. Just extend the domain of Δ to include sequences in ω^ω.

1.4 Null and Meager sets

If we are working with the Real line in the usual sense (Dedekind cuts or something similar), we say an open interval (a, b) has length $b - a$. If we are working in the Cantor (or Baire) space, and p is an element of $2^{<\omega}$ (or $\omega^{<\omega}$), then we say the open set U_p has length $2^{-\mathrm{dom}(p)}$.

Definition. The *outer measure* of a set $A \subseteq \mathbb{R}$ is defined by

$$\mu^*(A) := \inf \left\{ \sum_{i\in\omega} \mathrm{length}(I_i) : A \subseteq \bigcup_{i\in\omega} I_i \text{ and each } I_i \text{ is an open interval} \right\}$$

A set $B \subseteq \mathbb{R}$ is *Lebesgue measurable* if for each $A \subseteq \mathbb{R}$, $\mu^*(B) = \mu^*(B \cap A) + \mu^*(B \backslash A)$. If a set B is measurable, we write $\mu(B)$ instead of $\mu^*(B)$, and call this the *Lebesgue measure* of B.
A measurable set $B \subseteq \mathbb{R}$ is a *null set* if $\mu(B) = 0$.

We won't go any further in describing the Lebesgue measure. A complete description can be found in chapter 11 of [11]. We will however need a few basic properties.

Lemma 1.4.1. *Let $B \subseteq \mathbb{R}$ be a measurable set, then for each $\epsilon > 0$ there is a closed set V and an open set U such that $V \subseteq B \subseteq U$ and $\mu(U \backslash V) < \epsilon$.*

Proof. See [11]. □

Definition. A subset of $\mathbb{R}$ is a *G_δ-set* if it can be expressed as a countable intersection of open sets. A set subset of $\mathbb{R}$ is a *F_σ-set* if it can be expressed as a countable union of closed sets.

Corollary 1.4.2. *If $B \subseteq \mathbb{R}$ is a measurable set, then there is a F_σ set F and a G_δ set G such that $F \subseteq B \subseteq G$ and $\mu(F) = \mu(B) = \mu(G)$.*

Proof. For each $n \in \omega$ choose a closed set V_n, and an open set U_n such that $V_n \subseteq B \subseteq U_n$ and $\mu(U \backslash V) < \frac{1}{n}$. Then $F := \bigcup_{n\in\omega} V_n$ and $G := \bigcap_{n\in\omega} U_n$ are the desired sets. □

Definition. Let X be a set equipped with a topology, then a subset $A \subseteq X$ is called

- *dense* in X if for every open $U \subseteq X$, $U \cap A \neq \emptyset$.
- *nowhere dense* in X if there is a dense open set $U \subseteq X$ such that $U \cap A = \emptyset$.
- *meager* in X if it can be expressed as the union of countably many nowhere dense sets.

Lemma 1.4.3. *If $A \subseteq X$ is meager, there is a G_δ set B that can be expressed as the countable intersection of dense open sets such that $A \cap B = \emptyset$.*

Proof. This follows directly from the definitions of meager and nowhere dense. □

1.5 Some Universal Properties of the Real Line

We are going to want to work in several models of set theory, but want to be able to transfer some ideas like open sets from one model to another. For example, we may talk of the rational open interval with the *code* $(\frac{1}{2}, \frac{3}{4})$, but what we really want to talk about is the *interpretation* of the code, $\{x \in \mathbb{R} : \frac{1}{2} < x \wedge x < \frac{3}{4}\}$. We notice right away that the code is universal, as every model of set theory contains “the same” set of rational numbers, but as we shall soon see the interpretation may vary from model to model, because one model might have “more” real numbers between $\frac{1}{2}$ and $\frac{3}{4}$. We now extend this idea of *codes* to a few specific types of subsets of $\mathbb{R}$.

Definition. A set $p \in V$ is a *G-code* and represents an open set of $\mathbb{R}^V$ if it is an at most countable collection of ordered pairs of the form $\langle r_1, r_2\rangle$ where $r_1, r_2 \in \mathbb{Q}$ and $r_1 < r_2$. The *G-interpretation* of p in V is given by $p^V := \bigcup_{\langle r_1,r_2\rangle\in p}(r_1, r_2)^V$.
As above we define, $(r_1, r_2)^V := \{x \in \mathbb{R}^V : r_1 < x \wedge x < r_2\}$

Corollary 1.5.1. *If $q \in V$ is an open subset of $\mathbb{R}^V$, then there is a G-code $p \in V$ such that $p^V = q$.*

Proof. Every open subset of the real line is the union of countably many open rational intervals. □

Definition. A set $p \in V$ is an *F-code* and represents a closed set of $\mathbb{R}^V$ if it has the same form as a G-code. The *F-interpretation* of p in V however is given by $p^V := \mathbb{R}^V \setminus \bigcup_{\langle r_1,r_2\rangle\in p}(r_1, r_2)$. i.e. the complement of the G-interpretation of p.

Corollary 1.5.2. *If $q \in V$ is a closed subset of $\mathbb{R}^V$, then there is an F-code $p \in V$ such that $p^V = q$.*

Proof. Every closed subset of the real line is the complement of an open subset of the real line. □

Definition. A set $p \in V$ is a G_δ*-code* and represents a G_δ set of $\mathbb{R}^V$, if it is an at most countable collection of G-codes. The G_δ-interpretation of p in V is given by $p^V := \bigcap_{q\in p} q^V$. Where q^V is the G-interpretation of q in V.

We don't have to stop there. It is easy to see how one would define an F_σ code. We could even define codes for Borel sets. However, for us, as we shall later see, open, closed and G_δ sets will suffice.

Now that we have a way to code certain types of sets, it becomes interesting to ask which properties of these sets are universal. It turns out that the diameter of certain sets does not depend on the model we are working in. We will later be interested in the diameter of closed sets, so we will prove it for F-codes, but it would be easy to show this for other codes as well.

Lemma 1.5.3. *If $p \in V$ is an F-code, then $diam(p^V)$ depends only on p, and not on the interpretation p^V.*

Proof. We can show this with a simple calculation:

$$\begin{aligned}
\text{diam}(p^V) &= \sup\{x - y : x, y \in p^V\} \\
&= \sup\{x - y : \forall\langle r, s\rangle \in p(x \leq r \vee x \geq s) \wedge (y \leq r \vee y \geq s)\} \\
&= \sup\{x : \forall\langle r, s\rangle \in p(x \leq r \vee x \geq s)\} - \inf\{y : \forall\langle r, s\rangle \in p(y \leq r \vee y \geq s)\} \\
&= \sup\{r : \exists s\langle r, s\rangle \in p \wedge \forall\langle r', s'\rangle \in p(r \leq r' \vee r \geq s')\} \\
&\qquad\qquad -\inf\{s : \exists r\langle r, s\rangle \in p \wedge \forall\langle r', s'\rangle \in p(s \leq r' \vee s \geq s')\}
\end{aligned}$$

This term only depends on the code p, and not on the interpretation p^V. □

1.6 Cichoń's Diagram

The next natural question to ask, is which properties of the real line actually do depend on the model we are working in. For some specific examples, we will look at the 10 cardinal invariants that make up Cichoń's Diagram. First we define them.

Definition. A set $\mathcal{I} \subseteq \mathcal{P}(\mathbb{R})$ is *an ideal* on $\mathbb{R}$ if the following hold:

1. If $A, B \in \mathcal{I}$ then $A \cup B \in \mathcal{I}$
2. If $A \in \mathcal{I}$ and $B \subseteq A$ then $B \in \mathcal{I}$
3. If $r \in \mathbb{R}$ then $\{r\} \in \mathcal{I}$
4. $\mathbb{R} \notin \mathcal{I}$

Definition. If $\mathcal{I}$ is an ideal on $\mathbb{R}$, then define the following cardinal characteristics:

1. $\text{add}(\mathcal{I}) = \min\{card(\mathcal{A}) : \mathcal{A} \subseteq \mathcal{I}, \bigcup \mathcal{A} \notin \mathcal{I}\}$ is the *additivity number* of $\mathcal{I}$.
2. $\text{cov}(\mathcal{I}) = \min\{card(\mathcal{A}) : \mathcal{A} \subseteq \mathcal{I}, \bigcup \mathcal{A} = \mathbb{R}\}$ is the *covering number* of $\mathcal{I}$.
3. $\text{non}(\mathcal{I}) = \min\{card(X) : X \subseteq \mathbb{R}, X \notin \mathcal{I}\}$ is the *uniformity number* of $\mathcal{I}$.
4. $\text{cof}(\mathcal{I}) = \min\{card(\mathcal{A}) : \mathcal{A} \subseteq \mathcal{I}\ \forall B \in \mathcal{I}\ \exists A \in \mathcal{A}(B \subseteq A)\}$ is the *cofinality* of $\mathcal{I}$.

Definition. Let $\mathcal{N}$ denote the ideal of null sets, $\{X \subseteq \mathbb{R} : \mu(X) = 0\}$, and let $\mathcal{M}$ denote the ideal of meager sets.

Definition. If $f, g \in \omega^\omega$ then:

$$f \leq_n^* g \iff \forall m > n \ f(m) \leq g(m)$$

$$f \leq^* g \iff \exists n \in \omega \ f \leq_n^* g$$

If $f \leq^* g$, we say g *eventually dominates* f.

Definition. The *bounding number* is the smallest cardinality of an unbounded family on ω^ω with respect to $\leq^*$, defined by $\mathfrak{b} := \min\{\mathrm{card}(F) : F \subseteq \omega^\omega \text{ and } \forall g \in \omega^\omega \exists f \in F \text{ such that } f \not\leq^* g\}$.

Definition. The *dominating number* is the smallest cardinality of a cofinal family with respect to $\leq^*$, defined by $\mathfrak{d} = \min\{\mathrm{card}(F) : F \subseteq \omega^\omega \text{ and } \forall g \in \omega^\omega \exists f \in F \text{ such that } g \leq^* f\}$

Definition. *Cichoń's diagram* is the diagram:

$$\begin{tikzcd}
& \mathrm{cov}(\mathcal{N}) \arrow[r] & \mathrm{non}(\mathcal{M}) \arrow[r] & \mathrm{cof}(\mathcal{M}) \arrow[r] & \mathrm{cof}(\mathcal{N}) \arrow[r] & 2^{\aleph_0} \\
& & \mathfrak{b} \arrow[u] \arrow[r] & \mathfrak{d} \arrow[u] & & \\
\aleph_1 \arrow[r] & \mathrm{add}(\mathcal{N}) \arrow[r] \arrow[uu] & \mathrm{add}(\mathcal{M}) \arrow[r] \arrow[u] & \mathrm{cov}(\mathcal{M}) \arrow[r] \arrow[u] & \mathrm{non}(\mathcal{N}) \arrow[uu] &
\end{tikzcd}$$

It is well known that arrows in the diagram represent inequalities between these invariants that do not depend on the model. Most of these require long involved proofs, so we will omit them here. For a full explanation see [2]. Assuming the inequalities shown hold, it is clear that if we have the continuum hypothesis that we actually have equality everywhere. Later we will show a model where $\mathrm{cov}(\mathcal{N})$ and $\mathrm{cov}(\mathcal{M})$ are both $\aleph_2$, but $\mathfrak{b}$ is still $\aleph_1$. For more models see 7.5 in [3].

2

The Notion of Forcing

In 1963 Paul Cohen showed that both the axiom of choice and the continuum hypothesis are independent from the Zermelo-Fraenkel axioms of set theory. While his results were important in mathematics, it was his method that gave mathematicians a tool that could be used to prove much more. Best summarized by [9]: "The extent and breadth of the expansion of set theory henceforth dwarfed all that came before, both in terms of the numbers of people involved and the results established." The idea was simple: take an existing model of set theory and adjoin a new element, much like algebraists do when adding a transcendental to a field. In doing so, Cohen wanted to preserve the axioms of set theory, and also be careful not to add any new ordinals [6]. He decided to start with a countable transitive model, which gave insight on what a "generic" element might look like. Since the natural numbers are absolute in any model, the simplest set to try to add is a new subset of the them. Since the model was countable however, there exists an ordinal α, that is countable, but does not lie in the model. Since this α is countable, it could be encoded as a set of natural numbers. If we happen to try to adjoin this particular set, we would also add this ordinal α. So these "generic" elements had to be chosen wisely, and this is what the method of forcing does. We will present forcing here by using partial orders in the ground model, and show how to use them to find the sought after "generic" element.

2.1 The Definition of Forcing

Definition. Let V be a transitive model of set theory, and $(\mathbb{P}, \leq, 1) \in V$ a partially ordered set in which $1 \in \mathbb{P}$ is a maximal element. Then we will call $\mathbb{P}$ a *forcing* or a *forcing notion*, and its elements *forcing conditions*. We say p is *stronger* than q, or p is an *extension* of q, if $p \leq q$. We say p is *compatible* with q if there exists an r that extends both p and q, and we may denote this by $p \parallel q$. Otherwise p and q are called *incompatible*, and we may denote this by $p \perp q$.

Definition. If $(\mathbb{P}, \leq)$ is a partially ordered set, then $D \subseteq \mathbb{P}$ is *dense* in $\mathbb{P}$ if for every $p \in \mathbb{P}$ there is a $q \in D$ such that $q \leq p$.

Definition. A class $F \subseteq \mathbb{P}$ is a *filter* if

1. F is nonempty
2. If $p \leq q$ and $p \in F$ then $q \in F$
3. If $p, q \in F$ then there is an $r \in F$ such that $r \leq p, q$

A filter is $\mathbb{P}$-*generic* over V if

4. For all $D \in V$ such that D is dense in $\mathbb{P}$, $D \cap F \neq \emptyset$

Remark. We can actually drop the requirement that the ordering is antisymmetric, and take our forcing notions to be preorders. In any case, we can always take an appropriate quotient of a preorder to convert it to a partial order.

From now on, unless stated otherwise, we will assume that V is a transitive model of ZFC, $\mathbb{P} \in V$ is a forcing, and G is a $\mathbb{P}$-generic filter over V.

2.2 Properties of Generic Filters

In the definition of a generic filter, we use dense sets to describe its genericity. We could however use one of the following notions.

Definition. A set $D \subseteq \mathbb{P}$ is

- *predense* in $\mathbb{P}$ if for every $p \in \mathbb{P}$, there is some $q \in D$ and $r \in \mathbb{P}$ such that r extends both q and p.
- an *antichain* if for all $p, q \in A, p \perp q$. An antichain is *maximal* if for all $p \in \mathbb{P} \backslash A$, $A \cup \{p\}$ is not an antichain.

Lemma 2.2.1. *Let G be a filter on $\mathbb{P}$. Then the following are equivalent.*

1. *G is $\mathbb{P}$-generic over V*
2. *$G \cap D \neq \emptyset$ for every $D \in V$ that is predense in $\mathbb{P}$*
3. *$G \cap A \neq \emptyset$ for every $A \in V$ that is a maximal antichain in $\mathbb{P}$*

Proof. ($1 \implies 2$) If D is predense in $\mathbb{P}$, it is clear that $D' := \{p \in \mathbb{P} : \exists q \in D\ (p \leq q)\}$ is dense. Therefore if G is a generic filter, there is some $p \in D' \cap G$. Then by the definition of D', there is some $q \in D$ such that $p \leq q$. By the filter properties, $q \in G$, and therefore $q \in G \cap D$.

($2 \implies 3$) If A is a maximal antichain, and $p \in \mathbb{P}$, there is some $q \in A$ such that $p \parallel q$, (because otherwise $A \cup \{p\}$ would be still an antichain, so A wouldn't be maximal). Then let r extend p and q, and notice that this shows that A is predense. Therefore 3 follows directly from 2.

($3 \implies 1$) We show that every dense subset of $\mathbb{P}$ contains a maximal antichain. Let D be dense in $\mathbb{P}$, and let A be an antichain maximal with the property $A \subseteq D$. Suppose now that there exists a $p \in \mathbb{P} \backslash A$ such that $A \cup \{p\}$ is an antichain. Then since D is dense, there exists a $q \in D$ such that $q \leq p$. But then $A \cup \{q\}$ is an antichain and is contained in D, thus contradicting the assumed maximality of A. It follows that 3 implies 1. $\square$

Lemma 2.2.2. *If $\mathbb{P} \in V$ is a forcing, H is a filter on $\mathbb{P}$, and G is a $\mathbb{P}$-generic filter over V such that $G \subseteq H$, then $G = H$.*

Proof. For contradiction, assume this is not the case and let $p \in H$ be such that $p \notin G$. Then define the set $D := \{q \in \mathbb{P} : (q \leq p) \vee (q \perp p)\}$, and notice it is dense in $\mathbb{P}$. It follows from the genericity of G that we have some $q \in G \cap D$. Since q is an element of G but p is not, we can't have $q \leq p$, so it must be that $q \perp p$. Remember that $p \in H$, and notice that because $G \subseteq H$, $q \in H$. This of course is a contradiction though, as $p \perp q$. $\square$

Lemma 2.2.3. *Let $\mathbb{P} \in V$ be a forcing such that for every $p \in \mathbb{P}$ there exist $r, s \leq p$ such that $r \perp s$, and let $G \subseteq \mathbb{P}$ be a generic filter. Then $G \notin V$.*

Proof. We suppose for contradiction that $G \in V$. Then $\mathbb{P} \backslash G \in V$ and we can show that it is dense in $\mathbb{P}$. To do this, let $p \in \mathbb{P}$ and find $q, r \in \mathbb{P}$ such that $q, r \leq p$ and $q \perp r$. It cannot be the case that both q and r are elements of G, so either $q \in \mathbb{P} \backslash G$ or $r \in \mathbb{P} \backslash G$, thus $\mathbb{P} \backslash G$ is dense. By genericity $(\mathbb{P} \backslash G) \cap G \neq \emptyset$, but it is clear that this set would be empty, giving us a contradiction. □

While the above properties are important, the most important property of a generic filter is existence. Cohen's success of forcing came from the fact that he started with a countable minimal model. In such a model, one can ask every question in a sequence, and in particular find a subset of the forcing that intersects every dense subset in the ground model.

Lemma 2.2.4. *If M is a countable model, with a forcing $\mathbb{P} \in M$ and $p \in \mathbb{P}$ then there is a $\mathbb{P}$-generic filter, G, over M with $p \in G$.*

Proof. Start by enumerating all of the dense sets of $\mathbb{P}$ in M as $\langle D_n : n < \omega \rangle$ (this of course is done outside of M, as most likely this enumeration does not exist within M). By the density of D_0, choose a $p_0 \in D_0$ such that $p_0 \leq p$. Now for each $n \in \omega$ choose a $p_{n+1} \in D_{n+1}$ such that $p_{n+1} \leq p_n$, and take the filter generated by $\{p_n : n \in \omega\}$. By construction this filter is $\mathbb{P}$-generic over M, and contains p as an element. □

2.3 $\mathbb{P}$-names and Generic Extensions

Definition. An element $\dot{f} \in V$ is a $\mathbb{P}$-*name* if it is a relation and for all $\langle \dot{g}, p \rangle \in \dot{f}$ we have $\dot{g}$ is a $\mathbb{P}$-name and $p \in \mathbb{P}$.

It is clear from the definition that the empty set is trivially a $\mathbb{P}$-name. As we shall see below, there are "just as many" $\mathbb{P}$-names as there are elements in V.

Definition. If $x \in V$ then the *canonical* $\mathbb{P}$-*name* for x is defined recursively by

$$\check{x} := \{(\check{y}, 1) : y \in x\}$$

(In particular $\check{\emptyset} = \emptyset$.) Notice also that if $x \in V$ then $\check{x} \in V$ as well.

Definition. If $\dot{f} \in V$ is a $\mathbb{P}$-name, then the *G-interpretation* of $\dot{f}$ is given by

$$\dot{f}^G := \{\dot{g}^G : \exists p \in G(\langle \dot{g}, p \rangle \in \dot{f}\}$$

Now we can see why $\check{x}$ is called the canonical $\mathbb{P}$-name of x, as with a little work it is clear that $\check{x}^G = x$. Remember though that we have shown that G is not always an element of V. Thus we can see that the interpretation of some names may also not lie in V.

Definition. Even though the filter G may not be in V, it has a *canonical name* in V, namely

$$\dot{G} := \{(\check{p}, p\} : p \in \mathbb{P}\}$$

Notice that in fact $\dot{G}^G = G$, because if $p \in G$, then $p = \check{p}^G \in \dot{G}^G$, and if $p \notin G$, $p \notin \dot{G}^G$.

Definition. If G is a $\mathbb{P}$-generic filter over V, then define the $\mathbb{P}$-*generic extension* of V by G as

$$V[G] := \{\dot{f}^G : \text{“}\dot{f} \in V \text{ is a } \mathbb{P}\text{-name”}\}$$

We claim now that $V[G]$ is a model of ZFC. For a detailed proof, please refer to Kunen [10] chapter VII §4.

Lemma 2.3.1. *If $\tilde{V}$ is a transitive model of ZFC such that $V \subseteq \tilde{V}$ and $G \in \tilde{V}$ then $V[G] \subseteq \tilde{V}$.*

Proof. Since $G \in \tilde{V}$ and for each $\mathbb{P}$-name $\dot{f} \in V$, we also have $\dot{f} \in \tilde{V}$, it is clear that $\dot{f}^G \in \tilde{V}$. □

So one might think of $V[G]$ as being the smallest model of ZFC such that $V \subseteq V[G]$ and $G \in V[G]$. We can use this idea to define what a minimal generic extension would look like.

Definition. An extension $V[G]$ is *minimal* over V if given any subset of ordinals $X \in V[G]$, either $X \in V$ or $G \in V[X]$.

Since we are going to be working with codes for open sets, closed sets,... etc, the following lemma will become very useful to us. We will only present the case for G-codes (open sets), but it is clear that it can be repeated for any type of code.

Lemma 2.3.2. *Let $p \in V$ be a G-code for an open set $p^V \in V$, then $p^V \subseteq p^{V[G]}$.*

Proof. For each $x \in p^V$, there is some $\langle a, b\rangle \in p$ such that in V, $a \leq x \leq b$. This inequality holds in $V[G]$, and thus $x \in p^{V[G]}$. □

2.4 The Language of Forcing

Definition. We say $\phi(\dot{x}_0, ..., \dot{x}_n)$ is a *sentence in the forcing language* if ϕ is an $\in$-formula, and $\dot{x}_0, ..., \dot{x}_n \in V$ are $\mathbb{P}$-names.

Definition. If $\phi(\dot{x}_0, ..., \dot{x}_n)$ is a sentence in our forcing language, and p is a condition, we say p *forces* $\phi(\dot{x}_0, ..., \dot{x}_n)$ if for every $\mathbb{P}$-generic filter G with $p \in G$, the statement $\phi^{V[G]}(\dot{x}_0^G, ..., \dot{x}_n^G)$ is true in $V[G]$. We write in shorthand $p \Vdash \phi$.

Theorem 2.4.1. ***(The Forcing Theorem)***
If ϕ is a sentence of our forcing language then for every $\mathbb{P}$-generic $G \subseteq \mathbb{P}$ over V, we have:

$$V[G] \models \phi \iff \exists p \in G \ (p \Vdash \phi)$$

Proof. See theorem 3.6 in chapter VII §3 of [10]. □

Lemma 2.4.2. ***(Properties of Forcing)***
If ϕ and ψ are sentences in the forcing language, and p and q are conditions in $\mathbb{P}$ then

1. *If $p \Vdash \phi$ and $q \leq p$ then $q \Vdash \phi$*
2. *No p forces ϕ and $\neg\phi$*
3. *For every p there is a $q \leq p$ that decides ϕ, i.e. either $q \Vdash \phi$ or $q \Vdash \neg\phi$*
4. *$p \Vdash \phi$ if and only if no $q \leq p$ forces $\neg\phi$*
5. *$p \Vdash \phi \wedge \psi$ if and only if $p \Vdash \phi$ and $p \Vdash \psi$*
6. *$p \Vdash \forall x \ (\phi(x))$ if and only if $p \Vdash \phi(\dot{x})$ for every $\mathbb{P}$-name $\dot{x} \in V$*
7. *$p \Vdash \phi \vee \psi$ if and only if $\forall q \leq p \ (\exists r \leq q \ (r \Vdash \phi$ or $r \Vdash \psi))$*
8. *$p \Vdash \exists x \ (\phi(x))$ if and only if $\forall q \leq p \ (\exists r \leq q \exists \dot{x} \in V \ (r \Vdash \phi(\dot{x})))$*

Proof. Also found in chapter VII §3 of [10]. □

Some Consequences of the Forcing Theorems

Definition. A forcing $\mathbb{P}$ is *separative* if whenever $p \not\leq q$ there is a $p' \leq p$ such that $p' \perp q$.

Lemma 2.4.3. *If $\mathbb{P}$ is separative and $p, q \in \mathbb{P}$ then $p \Vdash \check{q} \in \dot{G}$ if and only if $p \leq q$.*

Proof. If $p \leq q$ and $p \in G$ then by the second filter property, $q \in G$. It is clear that by definition $p \Vdash \check{q} \in \dot{G}$. Now suppose that it is not the case that $p \leq q$. Since $\mathbb{P}$ is separative, we can find a $p' \leq p$ such that $p' \perp q$. It is clear by the forcing properties then that $p \nVdash \check{q} \in \dot{G}$, because $p' \Vdash \check{q} \notin \dot{G}$. $\square$

Lemma 2.4.4. *If $\dot{x} \in V$ is a $\mathbb{P}$-name such that $1 \Vdash \dot{x} \subseteq V$, then there is a name $\hat{x} \in V$ such that $1 \Vdash \dot{x} = \hat{x}$ and if $a \in \hat{x}$ then there is some $y \in V$ and $p \in \mathbb{P}$ such that $a = \langle \check{y}, p \rangle$.*

Proof. Define $\hat{x} := \{\langle \check{y}, p \rangle : y \in V \wedge p \in \mathbb{P} \wedge (p \Vdash \check{y} \in \dot{x})\}$, then we claim that $1 \Vdash \dot{x} = \hat{x}$. First let G be a $\mathbb{P}$-generic filter over V. Suppose that $y \in \dot{x}^G$ and notice that by the forcing theorem there is some $p \in G$ such that $p \Vdash \check{y} \in \dot{x}$, which by definition tells us that $\langle \check{y}, p \rangle \in \hat{x}$. Therefore $p \Vdash \check{y} \in \hat{x}$, and in particular since $p \in G$, $y = \check{y}^G \in \hat{x}^G$. Now suppose instead that $y \in \hat{x}^G$, and notice again there is a $p \in G$ such that $p \Vdash \check{y} \in \hat{x}$. Then there is some $q \geq p$ such that $\langle \check{y}, q \rangle \in \hat{x}$ (in particular we could show this to be true for $q = p$), so $q \Vdash \check{y} \in \dot{x}$. Since $p \in G$ and $q \geq p$, we have $q \in G$, so $y = \check{y}^G \in \dot{x}^G$. Together, we get $\dot{x}^G = \hat{x}^G$, or because G was an arbitrary generic filter, $1 \Vdash \dot{x} = \hat{x}$. $\square$

Lemma 2.4.5. *Suppose that $A \in V$ is an antichain in $\mathbb{P}$, and for each $q \in A$, $\dot{x}_q \in V$ is a $\mathbb{P}$-name. Then there is a $\dot{x} \in V$ such that $q \Vdash \dot{x} = \dot{x}_q$ for each $q \in A$.*

Proof. Define $\dot{x} := \bigcup_{q \in A} \{\langle \dot{y}, p \rangle : \dot{y} \in \mathrm{dom}(\dot{x}_q) \wedge p \leq q \wedge p \Vdash \dot{y} \in \dot{x}_q\}$. Fix a $q \in A$ and assume G is a generic filter with $q \in G$. We now show that $\dot{x}^G = \dot{x}_q^G$.
Let $y \in \dot{x}^G$. Then there exists an $r \in A \cap G$, a $p \in G$, with $p \leq r$, and a name $\dot{y} \in \mathrm{dom}(\dot{x}_r)$ such that $\langle \dot{y}, p \rangle \in \dot{x}$, $\dot{y}^G = y$, and $p \Vdash \dot{y} \in \dot{x}_r$. Since A is an antichain, the only element of $A \cap G$ is q, and since $p \in G$, $\dot{y}^G \in \dot{x}_q^G$.
Now let $y \in \dot{x}^G$. Then there exists an $p \in G$ with $p \leq q$ and a $\dot{y} \in \mathrm{dom}(\dot{x})$ such that $p \Vdash \dot{y} \in \dot{x}$. By the definition of $\dot{x}$ it must be that $\dot{y} \in \dot{x}_q$, and $p \Vdash \dot{y} \in \dot{x}_q$. It is then clear that $\dot{y}^G \in \dot{x}_q^G$. $\square$

Theorem 2.4.6. ***(The Maximality Principal)***
(Assume AC holds in V) Let $q \in \mathbb{P}$, and suppose that for some $\mathbb{P}$-names, $\dot{x}_1, ..., \dot{x}_n \in V$, we have $q \Vdash \exists x\ (\phi(x, \dot{x}_1, ..., \dot{x}_n)$. Then there is a $\mathbb{P}$-name $\dot{x} \in V$ such that $q \Vdash \phi(\dot{x}, \dot{x}_1, ..., \dot{x}_n)$.

Proof. Using Zorn's Lemma in V, find an antichain $A \in V$ maximal with the properties:

1. A is an antichain in $\mathbb{P}$
2. $\forall p \in A(p \leq q \wedge \exists \dot{x} \in V\ (p \Vdash \phi(\dot{x}, \dot{x}_1, ..., \dot{x}_2)))$

Using the axiom of choice, choose for each $p \in A$ a name $\dot{x}_p \in V$ such that $p \Vdash \phi(\dot{x}_p, \dot{x}_1, ..., \dot{x}_2)$, and by the previous lemma, choose an $\dot{x} \in V$ such that $p \Vdash \dot{x} = \dot{x}_p$ for each $p \in A$. Assume now that $q \nVdash \phi(\dot{x}, \dot{x}_1, ..., \dot{x}_n)$, then there is some $r \leq q$ such that $r \Vdash \neg\phi(\dot{x}, \dot{x}_1, ..., \dot{x}_n)$. Since $r \leq q$, $r \Vdash \exists x(\phi(x, \dot{x}_1, ..., \dot{x}_n)$, so there is some $s \leq r$ and some $\mathbb{P}$-name $\dot{y}$ such that $s \Vdash \phi(\dot{y}, \dot{x}_1, ..., \dot{x}_n)$. Since $s \Vdash \neg\phi(\dot{x}, \dot{x}_1, ..., \dot{x}_n)$, we can see that $s \perp p$ for every $p \in A$, but then $A \cup \{s\}$ is an antichain in V with the desired properties but $A \subsetneq A \cup \{s\}$, contradicting the maximality of A. $\square$

2.5 Equivalence of Forcing Notions

We are going to look at a few specific examples of forcing notions that add new real numbers to the ground model. Sometimes two different forcing notions produce the same generic extension. We can use this to our advantage by producing multiple forcing notions, with different combinatorial properties, that produce the same generic extension. Some specific properties of the generic extension might be easier to show in one variation over another. We first describe here a sufficient condition that two forcing notions are equivalent.

Definition. Two forcing notions in a ground model V are *equivalent* if they generate the same generic extensions of V.

Definition. Let $\mathbb{P}$ and $\mathbb{Q}$ be forcing notions in our ground model V, then a map $f \in V$, $f : \mathbb{P} \to \mathbb{Q}$ is called a *dense embedding* if:

1. If $p_1 \leq p_2$ then $f(p_1) \leq f(p_2)$
2. If p_1 and p_2 are incompatible then so are $f(p_1)$ and $f(p_2)$
3. $f[\mathbb{P}]$ is dense in $\mathbb{Q}$

Theorem 2.5.1. *If $f : \mathbb{P} \to \mathbb{Q}$ is a dense embedding then $\mathbb{P}$ and $\mathbb{Q}$ are equivalent forcing notions.*

Proof. As dense sets play an important role in genericity, we need a way to translate them from one forcing to the other. We do this with the following two claims.

Claim. *If $D \in V$ is dense in $\mathbb{P}$, then $f[D]$ is dense in $\mathbb{Q}$.*

Proof. Let $q \in \mathbb{Q}$, then since $f[\mathbb{P}]$ is dense in $\mathbb{Q}$ we can find a $p' \in \mathbb{P}$ such that $f(p') \leq q$, and since D is dense in $\mathbb{P}$ we can find $p \in D$ such that $p \leq p'$. Then $f(p) \in f[D]$ and $f(p) \leq f(p') \leq q$, so $f[D]$ is dense in $\mathbb{Q}$. □

To transfer dense sets in the other direction, we can't just take the preimage of D because it might not have one. To get around this problem, we take the set of everything that "would be under" the preimage.

Claim. *If $D \in V$ is dense in $\mathbb{Q}$, then $D_{\mathbb{P}} := \{p \in \mathbb{P} : \exists q \in D(f(p) \leq q)\}$ is dense in $\mathbb{P}$.*

Proof. Let $p \in \mathbb{P}$, then $f(p) \in \mathbb{Q}$, and D is dense in $\mathbb{Q}$, so find a $d \in D$ such that $d \leq f(p)$. Now, since $f(\mathbb{P})$ is also dense in $\mathbb{Q}$ find a $p' \in \mathbb{P}$ such that $f(p) \leq d$. It is clear that $f(p') \leq f(p)$, so in particular they are compatible, which means by property 2. of the dense embedding, so are p and p'. Thus we can find a $p'' \in \mathbb{P}$ that extends both p and p'. By property 1. of the embedding, $f(p'') \leq f(p') \leq d$, and thus $p'' \in D_{\mathbb{P}}$, proving that it is dense in $\mathbb{P}$. □

Now we want to construct some maps (computable in the appropriate universes) that translate generic filters from one forcing to the other.

First we define a map $\bar{f}$ such that if $G \subseteq \mathbb{P}$ is a $\mathbb{P}$-generic filter over V, the image $\bar{f}(G) \subseteq \mathbb{Q}$ is a $\mathbb{Q}$-generic filter over V. Notice that the image of a filter might not be upwards closed, and therefore not a filter. It turns out that the right thing to do is to take the upwards closure of this image.

Claim. *Let $G \subseteq \mathbb{P}$ be a $\mathbb{P}$-generic filter over V, then $\bar{f}(G) := \{q \in \mathbb{Q} : \exists p \in G(f(p) \leq q)\} \subseteq \mathbb{Q}$ is a $\mathbb{Q}$-generic filter over V.*

Proof. First we show that $\bar{f}(G)$ is a filter on $\mathbb{Q}$.

1. $\bar{f}(G)$ is not empty

   This is trivial, because G is not empty.

2. If $p \leq q$ and $p \in \bar{f}(G)$, then $q \in \bar{f}(G)$

   Since $p \in \bar{f}(G)$, there is some $r \in G$ such that $f(r) \leq p$. It is then clear that $f(r) \leq q$, so $q \in \bar{f}(G)$.

3. If $p, q \in \bar{f}(G)$ then there is an $r \in \bar{f}(G)$ such that $r \leq p, q$

   Let $p', q' \in G$ such that $f(p') \leq p$ and $f(q') \leq q$, then because G is a filter, there is some $r \leq p', q'$ such that $r \in G$. We can see now that $f(r) \in \bar{f}(G)$ and $f(r) \leq p, q$.

Now we will show that the filter $\bar{f}(G)$ is $\mathbb{Q}$-generic over V.

4. Given any set $D \in V$ which is dense in $\mathbb{Q}$, $\bar{f}(G) \cap D \neq \emptyset$.

   Since D is dense in $\mathbb{Q}$, the set $D_{\mathbb{P}}$ defined in our claim above is dense in $\mathbb{P}$, and thus by the $\mathbb{P}$-genericity of G, there is some $p \in G \cap D_{\mathbb{P}} \neq \emptyset$. Since $p \in D_{\mathbb{P}}$ there is a $q \in D$ such that $f(p) \leq q$, and because $p \in G$, by definition $q \in \bar{f}(G)$.

□

Now we show that in taking the inverse image under f of a $\mathbb{Q}$-generic filter H, the result is a $\mathbb{P}$-generic filter.

Claim. *Let $H \subseteq \mathbb{Q}$ be a $\mathbb{Q}$-generic filter over V, then $f^{-1}[H] := \{p \in \mathbb{P} : f(p) \in H\} \subseteq \mathbb{P}$ is a $\mathbb{P}$-generic filter over V.*

Proof. Again we begin by showing that the result is a filter.

1. $f^{-1}[H]$ is not empty

   Since $f[\mathbb{P}] \in V$ is dense in $\mathbb{Q}$, and H is a $\mathbb{Q}$-generic filter over V, $f[\mathbb{P}] \cap H \neq \emptyset$. Thus if $p \in \mathbb{P}$ such that $f(p) \in H$, it is clear $p \in f^{-1}[H]$.

2. If $p \leq q$ and $p \in f^{-1}[H]$, then $q \in f^{-1}[H]$

   By property *1.* of f, $f(p) \leq f(q)$, and thus by the filter properties of H, $f(q) \in H$, so $q \in f^{-1}[H]$.

3. If $p, q \in f^{-1}[H]$ then there is an $r \in f^{-1}[H]$ such that $r \leq p, q$

   Define the set $D_{p,q} := \{r \in \mathbb{P} : (r \leq p, q) \vee (r \perp p) \vee (r \perp q)\}$. This is clearly dense in $\mathbb{P}$, since if we have any $s \in \mathbb{P}$, it is either compatible with both p and q, or incompatible with one of them. If it is incompatible with one of them, it is already an element of $D_{p,q}$. Otherwise it is compatible with both, so we can choose some r stronger than p, q and s, and this is an element of $D_{p,q}$. We know that $f[D_{p,q}]$ is dense in $\mathbb{Q}$ by the claim above, so $H \cap f[D_{p,q}] \neq \emptyset$. Let $r \in D_{p,q}$ such that $f(r) \in H \cap f[D_{p,q}]$. Since $f(r) \parallel f(p), f(q)$, (as they are all elements of the filter H), r must be compatible with both p and q by property *2.* of the map f. By the definition of $D_{p,q}$ this means that $r \leq p, q$, and is thus the desired element of $f^{-1}[H]$.

Now we will show that the filter $f^{-1}[H]$ is $\mathbb{P}$-generic over V.

4. Given any set $D \in V$ which is dense in $\mathbb{P}$, $f^{-1}[H] \cap D \neq \emptyset$.

   We know already that $f[D] \in V$ is dense in $\mathbb{Q}$, so $f[D] \cap H \neq \emptyset$. Let $r \in D$ be such that $f(r) \in f[D] \cap H$, then it is clear that $r \in f^{-1}[H] \cap D$.

$\square$

We now want to show that these operations are in a way "inverses" of each other. We do so with the following two claims.

Claim. *If $G \subseteq \mathbb{P}$ is a $\mathbb{P}$-generic filter over V, then $f^{-1}[\bar{f}(G)] = G$.*

Proof. It is clear that $G \subseteq f^{-1}[\bar{f}(G)]$, because if $p \in G$, we have $p \in f^{-1}[\{f(p)\}] \subseteq G'$. We have already shown that both are $\mathbb{P}$-generic filters, so equality follows from **lemma 2.2.2**.

$\square$

Claim. *If $H \subseteq Q$ is a $\mathbb{Q}$-generic filter over V, then $H' := \bar{f}(f^{-1}[H]) = H$.*

Proof. It is clear that $H' \subseteq H$, because if $q \in \bar{f}(f^{-1}[H])$, there is some $p \in f^{-1}[H]$ (so $f(p) \in H$) such that $f(p) \leq q$, which means $q \in H$. Again equality follows from **lemma 2.2.2**.

$\square$

So here we have shown that by using f we can take a $\mathbb{P}$-generic filter, translate it to a $\mathbb{Q}$-generic filter, and can recover the original filter from this translation. We can also do the same thing starting with a $\mathbb{Q}$-generic filter. Thus we have shown that $\mathbb{P}$ and $\mathbb{Q}$ generate the same generic extensions of V.

$\square$

2.6 Preserving Cardinals

Definition. Let $\alpha \in V$ be a cardinal. Then we say a forcing $\mathbb{P}$ satisfies the *α-chain condition* if every antichain has cardinality less than α. The $\aleph_1$-chain condition may also be referred to as the *countable chain condition* or simply ccc.

Lemma 2.6.1. *Let $\alpha \in V$ be a cardinal and assume that $\mathbb{P}$ satisfies the α-chain condition, then*

1. *$\mathbb{P}$ preserves cofinalities greater than or equal to α.*
2. *If α is regular in V, then $\mathbb{P}$ preserves all cardinals greater than or equal to α.*

In particular, if $\mathbb{P}$ satisfies the ccc, then $\mathbb{P}$ preserves all cardinals and cofinalities.

Proof. For the following, let G be a $\mathbb{P}$-generic filter over V.
1. Let β be a cardinal in V such that $\mathrm{cf}^V(\beta) \geq \alpha$. Let $\gamma \in \mathrm{Ord}$ and assume $f \in V[G]$ is a cofinal map $f : \gamma \to \beta$. Find a name $\dot{f} \in V$ for f, and a $p \in G$ such that $p \Vdash$ "$\dot{f}$ is a map from γ to β". Now for each $\xi < \gamma$ define in V the set $A_\xi := \{\delta < \beta : \exists q \leq p\ (q \Vdash \dot{f}(\check{\xi}) = \check{\delta})\}$. Since $\mathbb{P}$ has the α-chain condition we can show that $\mathrm{card}^V(A_\xi) < \alpha$. To do this, choose a witness q_δ for each $\delta \in A_\xi$ such that $q_\delta \Vdash \dot{f}(\check{\xi}) = \check{\delta}$. Then notice that $\{q_\delta : \delta \in A_\xi\}$ is an antichain of $\mathbb{P}$. It is easy to see that $\mathrm{range}(f) \subseteq \bigcup_{\xi<\gamma} A_\xi$, so in particular, this set is cofinal in β. Since $\bigcup_{\xi<\gamma} A_\xi \in V$, and $\mathrm{cf}^V(\beta) \geq \alpha$, we have $\gamma \geq \mathrm{cf}^V(\beta)$, and thus $\mathrm{cf}^V(\beta) = \mathrm{cf}^{V[G]}(\beta)$.

2. Assume that α is regular in V. If $\beta \geq \alpha$ is a regular cardinal in V, by *1.*, $\mathrm{cf}^{V[G]}(\beta) = \mathrm{cf}^V(\beta) = \beta$, so β is a regular cardinal in V. If $\beta > \alpha$ is a limit cardinal in V, then it is clear that the set $\{\gamma : \alpha \leq \gamma < \beta \wedge$ "γ is a regular cardinal of V"$\}$ is unbounded in β, and because each of these regular cardinals is still a regular cardinal in $V[G]$, β is also a cardinal in $V[G]$. $\square$

3

Examples of Generic Reals

Now that we have introduced the notion of forcing, we will give three specific examples that add new reals to the ground model, and examine some of their properties. Along the way, we will show that the first two forcing notions exhibit the countable chain condition, and even though the third does not, we will show that it does not collapse cardinals for another reason. All of our examples come from [7], but can also be found in [8], [13], [10] and many more.

3.1 Cohen Reals

As stated earlier, forcing was discovered by Paul Cohen in 1963 while proving the independence of the Axiom of Choice and of the Continuum Hypothesis from the Zermelo-Fraenkel axioms of set theory. In his papers [4] and [5] Cohen introduces the technique of forcing to prove the independence of the Continuum Hypothesis, and uses a forcing notion that will become the prototype of the "Cohen forcing" that we will now examine. We will define this forcing in three distinct ways, but show that all are equivalent.

Forcing with Finite Sequences of 0's and 1's

We define our forcing by $\mathbb{C}_2 := 2^{<\omega}$ (the set of all finite sequences of 0's and 1's), and say $p \in \mathbb{C}_2$ is stronger than $q \in \mathbb{C}_2$ if $p \supseteq q$. (i.e. The sequence q is an initial segment of p.) We notice that with this ordering, the empty sequence $\emptyset$ is our maximal element.

Forcing with Finite Sequences of Natural Numbers

We define another forcing $\mathbb{C}_\omega := \omega^{<\omega}$, saying again that $p \in \mathbb{C}_\omega$ is stronger than $q \in \mathbb{C}_\omega$ if $p \supseteq q$. Again, we see the empty sequence is the maximal element of this forcing.

Forcing with Partial Functions from ω to ω with finite domain

Yet another way that we could define the Cohen forcing is by letting the forcing conditions be partial functions $p : \mathrm{dom}(p) \subseteq \omega \to \omega$, where $\mathrm{card}(\mathrm{dom}(p)) < \omega$. Let us call this forcing $\mathbb{C}_f$, and say $p \in \mathbb{C}_f$ is stronger than $q \in \mathbb{C}_f$ if $p \supseteq q$. The maximal element of this forcing is the empty function.

Lemma 3.1.1. *The Cohen forcing notions $\mathbb{C}_2$, $\mathbb{C}_\omega$, and $\mathbb{C}_f$ defined above are all equivalent.*

Proof. First notice that $\mathbb{C}_\omega$ has a natural embedding into $\mathbb{C}_f$, because any finite sequence of natural numbers can be thought of as a function from its length to the natural numbers. Formally we can think of it as the subset set $\{f \in \mathbb{C}_f : \forall m, n \in \omega, (m < n \wedge n \in \text{dom}(f)) \rightarrow (m \in \text{dom}(f))\}$. This set is clearly dense, as given any function $f \in \mathbb{C}_f$, we can fill in the gaps in the domain to create the sequence $\bar{f} := f \cup \{(n, 0) : n \in \max(\text{dom}(f)) \backslash \text{dom}(f)\} \in \mathbb{C}_\omega$, and we have $\bar{f} \leq f$. Thus by **theorem 2.5.1**, $\mathbb{C}_\omega$ is equivalent to $\mathbb{C}_f$.

Notice that there is also an obvious embedding of $\mathbb{C}_2$ into $\mathbb{C}_\omega$, but the image is not dense, so we need a different approach.

We will show $\mathbb{C}_\omega$ is equivalent to $\mathbb{C}_2$ by defining the map $f : \mathbb{C}_\omega \rightarrow \mathbb{C}_2$ by the following. Define first $g : \omega \rightarrow 2^{<\omega}$ by $g(x) := \{(n, x_n) : n \leq x \wedge (n < x \rightarrow x_n = 1) \wedge (n = x \rightarrow x_n = 0\}$. i.e. $g(x)$ is the sequence of x 1's followed by a 0. Now define f recursively:

$$f(\emptyset) = \emptyset$$

If f is defined for $p \in \mathbb{C}_\omega$, and $x \in \omega$

$$f(p^\frown x) := f(p)^\frown g(x)$$

(e.g. $f(< 1, 0, 2, 4 >) =< \underbrace{1, 0,}_{1,} \underbrace{0,}_{0} \underbrace{1, 1, 0,}_{2} \underbrace{1, 1, 1, 1, 0}_{4} >$)

Remember that $p_1 \leq p_2$ in $\mathbb{C}_\omega$ if $p_2 \subseteq p_1$. It is clear however that $p_2 \subseteq p_1$ if and only if $f(p_2) \subseteq f(p_1)$. Now notice that $f(\mathbb{C}_\omega) = \{q \in \mathbb{C}_2 : \text{the last term in the sequence is } 0\}$, which is dense in $\mathbb{C}_2$ (because we can add a zero to the end of any finite sequence). We have shown this is a dense embedding, so by **theorem 2.5.1**, these forcing notions are equivalent. $\square$

Theorem 3.1.2. *The Cohen forcing notions have the countable chain condition.*

Proof. Each forcing described above is itself countable, so of course any antichain contained in one of them is also countable. $\square$

We claim now that these forcing notions give rise to a real number that is not available to us in the ground model. We will only show this explicitly for $\mathbb{C}_\omega$, but the proof is similar for the other two notions.

Theorem 3.1.3. *If G is a generic filter on $\mathbb{C}_\omega$, then $f := \bigcup G$ is a real number in the model $V[G]$.*

Proof. We will show that $f : \omega \rightarrow \omega$ is a function (and thus a real number). Notice that if p,$q \in G$ then either $p \subseteq q$ or $q \subseteq p$. Thus p and q agree on any common domain, so $f : dom(f) \rightarrow \omega$ is in fact a function. Next notice that the sets $D_n := \{p \in \mathbb{C}_\omega : dom(p) \geq n\}$ are dense in $\mathbb{C}_\omega$. By genericity $D_n \cap G \neq \emptyset$ (and thus $n \in \text{dom}(f)$) for every $n \in \omega$, so $dom(f) = \omega$. $\square$

It is clear that given any forcing condition, we can find two extensions of it that are incompatible, so by **lemma 2.2.3** we know $G \notin V$. We show that $f \notin V$ by the following lemma.

Lemma 3.1.4. *The generic filter can be recovered from f. In particular $V[f] = V[G]$*

Proof. We show now that $G = \{p \in \mathbb{C}_\omega : p \subseteq f\}$. One inclusion is obvious, as if $p \in G$, it is clear that $p \subseteq f$. To show the other direction, suppose that $f \supseteq p \in \mathbb{C}_\omega$ and for each $n \in \text{dom}(p)$ find a $p_n \in G$ such that $(n, p(n)) \in p_n$. Then by the filter properties (and because $\text{dom}(p)$ is finite), there is some $p' \in G$ such that for every $n \in \text{dom}(p)$, $p_n \subseteq p'$. Of course $p \subseteq p'$, so $p \in G$ because G is a filter. Now that G can be recovered from f, we have shown $V[f] = V[G]$. $\square$

Theorem 3.1.5. *In the Cohen extension $V[G]$ there are functions $f : \omega \to \omega$ that are not dominated by any function in V.*

Proof. Again, we will work with the forcing $\mathbb{C}_\omega$ but there are similar proofs for the other two notions as well. Let G be a generic filter on $\mathbb{C}_\omega$ and $f := \bigcup G$. Now let $g : \omega \to \omega$ be any function in V, and define $D_g := \{p \in \mathbb{C}_\omega : \exists n \in \operatorname{dom}(p)(g(n) < p(n))\}$. We can see that D_g is dense, because given any $q \in \mathbb{C}_\omega$ and $n \notin \operatorname{dom}(p)$, $p \cup \{(n, g(n) + 1)\} \leq p$. Thus let $d \in D_g \cap G$, and notice that $d \subseteq f$, so g does not dominate f. □

We would now like to show that that the Cohen extension $V[G]$ is not a minimal extension over V. In fact, we will show that there is an intermediate extension that is also a Cohen extension.

Theorem 3.1.6. *If G is a $\mathbb{C}_2$-generic filter over V, then there exist filters $G_0, G_1 \subseteq \mathbb{C}_2$ such that G_0 is $\mathbb{C}_2$-generic over V, G_1 is $\mathbb{C}_2$-generic over $V[G_0]$, and $V[G] = V[G_0][G_1]$. In particular $V[G]$ is not a minimal generic extension of V.*

Proof. For $f \in \mathbb{C}_2$, define new functions $f_0, f_1 \in \mathbb{C}_2$ by defining $f_0(n) := f(2n)$ and $f_1(n) := f(2n+1)$. Now define a map $F : \mathbb{C}_2 \to \mathbb{C}_2 \times \mathbb{C}_2$ by setting $F(f) := \langle f_0, f_1 \rangle$. If we let $\mathbb{C}_2 \to \mathbb{C}_2$ take the product ordering, it is not hard to show that this is a dense embedding, so there is a $\bar{G} \in \mathbb{C}_2 \times \mathbb{C}_2$ such that $V[G] = V[\bar{G}]$. Now if $\pi_i : \mathbb{C}_2 \times \mathbb{C}_2 \to \mathbb{C}_2$ is the canonical projection (i.e. $\pi_i(f_0, f_1) = f_i$) then define $G_i := \pi_i[\bar{G}]$. We claim that these are the desired filters.

Claim. *Both G_0 and G_1 are filters on $\mathbb{C}_2$.*

Proof.

1. $G_0 \neq \emptyset$
   This is clear, as $\bar{G} \neq \emptyset$.
2. If $p \leq q$ and $p \in G_0$ then $q \in G_0$
   Let $p' \in \mathbb{C}_2$ such that $\langle p, p' \rangle \in \bar{G}$. Then $\langle p, p' \rangle \leq \langle q, \emptyset \rangle$, so $\langle q, \emptyset \rangle \in \bar{G}$, and thus $q \in G_0$.
3. If $p, q \in G_0$ there is an $r \in G_0$ such that $r \leq p, q$
   Let $p', q' \in \mathbb{C}_2$ such that $\langle p, p' \rangle, \langle q, q' \rangle \in \bar{G}$, then there is some $\langle r, r' \rangle \in \bar{G}$ such that $\langle r, r' \rangle \leq \langle p, p' \rangle, \langle q, q' \rangle$. By definition $r \in G_0$ and $r \leq p, q$.

Similarly G_1 satisfies all of these. □

Claim. *G_0 is $\mathbb{C}_2$-generic over V.*

Proof. Let $D \in V$ be dense in $\mathbb{C}_2$. Then notice that $\pi_0^{-1}[D]$ is dense in $\mathbb{C}_2 \times \mathbb{C}_2$ by the following. Let $\langle p, q \rangle \in \mathbb{C}_2 \times \mathbb{C}_2$, then by the density of D, find a $p' \in \mathbb{C}_2$ such that $p' \leq p = \pi_0(\langle p, q \rangle)$. Now we have $\langle p', q \rangle \in \pi_0^{-1}[D]$, and $\langle p', q \rangle \leq \langle p, q \rangle$. So by genericity of $\bar{G}$ we can find a $\langle p, q \rangle \in \pi_0^{-1}[D] \cap \bar{G}$, which gives us $p \in D \cap G_0$. □

Claim. *G_1 is $\mathbb{C}_2$-generic over $V[G_0]$.*

Proof. Let $D \in V[G_0]$ be dense in $\mathbb{C}_2$, let $\dot{D} \in V$ be a name for D, and let $p_0 \in G_0$ such that $p_0 \Vdash$ "$\dot{D}$ is dense in $\check{\mathbb{C}}_2$". Now define $D' \in V$ by $D' := \{\langle q_0, q_1 \rangle : (q_0 \leq p_0) \wedge (q_0 \Vdash \check{q}_1 \in \dot{D}\}$, and show it is dense under $\langle p, \emptyset \rangle$ in $\mathbb{C}_2 \times \mathbb{C}_2$ by the following. Let $\langle p, q \rangle \in \mathbb{C}_2 \times \mathbb{C}_2$ be such that $p \leq p_0$. Now since $p \leq p_0$, $p \Vdash \exists x \in \check{\mathbb{C}}_2(x \in \dot{D} \wedge x \leq \check{q})$, so in particular, there is a $p' \leq p$ and $q' \in \mathbb{C}_2$ such that $p' \Vdash \check{q}' \in \dot{D} \wedge \check{q}' \leq \check{q}$. In other words $\langle p', q' \rangle \leq \langle p, q \rangle$ and $\langle p', q' \rangle \in D'$. Now that we know $D' \in V$ is dense below p_0 in $\mathbb{C}_2 \times \mathbb{C}_2$, we can find a $\langle p, q \rangle \in D' \cap \bar{G}$. So we have $q \in G_1$, and $p \in G_2$ such that $p \Vdash \check{q} \in \dot{D}$, and thus $q \in D \cap G_1$. □

It doesn't take much work to show that we can recover $\bar{G}$ from these two filters by $\bar{G} = G_0 \times G_1$, so the end result is $V[G] = V[\bar{G}] = V[G_0][G_1]$.

Notice that $A := \{n \in \omega : f_0(n) = 1\}$ is a subset of the natural numbers (and therefore a subset of the ordinals) and $A \in V[G]$, but $V[A] = V[G_0] \subsetneq V[G]$, so $V[G]$ is not minimal over V. $\square$

Lemma 3.1.7. *If d is a G_δ-code of the intersection of countably many open dense subsets of $\mathbb{R}$ in V, and $a = \bigcup G$ is the Cohen real generated by the forcing $\mathbb{C}_\omega$, then $a \in d^{M[G]}$.*

Proof. If $c \in d$, then c is a G_δ-code for a dense open set in V. Define now

$$D_c := \{p \in \mathbb{C}_\omega : \exists \langle r, s\rangle \in c \ \ \forall y \in \mathbb{R}(p \subseteq y \to y \in (r, s))\}$$

We show first that D_c is dense in $\mathbb{C}_\omega$. To do this, let $q \in \mathbb{C}_\omega$, and remember $U_q := \{x \in \omega^\omega : q \subseteq x\}$ is a basic open set in $\mathbb{R}$. Since c^M is dense in $\mathbb{R}^M$, and $(U_q)^M$ is an open subset of $\mathbb{R}^M$, there is some $y_0 \in (U_q)^M \cap c^M$, and thus some $\langle r, s\rangle \in c$ such that $y_0 \in (r, s)^M$. Since the sets U_p where $p \in \mathbb{C}_\omega$ generate a basis for the topology of $\mathbb{R}$, we can find some $p \leq q$ such that $U_p \subseteq (r, s)$, and thus $p \in D_c$. Now that we have a dense set, by genericity of G, let $p \in G \cap D_c$, and notice that $p \subseteq a$, and therefore $a \in (r, s)^{M[G]} \subseteq c^{M[G]}$. Since this holds for all $c \in d$, the result is $a \in d^{M[G]}$. $\square$

3.2 Random Reals and the Solovay Forcing

Soon after Cohen's introduction of forcing, Robert Solovay introduced the concept of the *random real*. This real is called "random" because, as we will show, given a name for any Borel set of full Lebesgue measure in the ground model, the new random real will appear as an element of the interpretation of said name in the generic extension. (By full measure, we mean a set whose complement has measure zero.) In 1965, while working with Lebesgue measurability, Solovay was led to assigning a truth value from a complete Boolean algebra to each formula [9]. He in in turn developed a method of forcing using complete Boolean algebras that was made popular by Dana Scott at a set theory conference held at UCLA in the summer of 1967. Scott even reproduced the work of Cohen using Boolean models in the paper [12]. We however will stay with forcing with partial orders, but will borrow the technique of assigning truth values to show an important property of random reals.

Forcing with Borel Sets

We let our forcing conditions be the Borel sets in $\mathbb{R}$ of positive Lebesgue measure. We then say $p \in \mathbb{B}$ is stronger than $q \in \mathbb{B}$ if $\mu(p \backslash q) = 0$. It may be easier to remember $p \leq q$ if $p \subseteq q$ *almost everywhere*. We can see that our maximal element can be the real line itself.

Forcing with Closed Sets

For convenience, we will focus our attention on the subforcing $\mathbb{B}_C$ defined by the closed (and therefore Borel) subsets of the real line with positive Lebesgue measure. The maximal element is again the full real line. This is clearly a dense embedding by the properties of Borel sets introduced in Chapter 1, and thus equivalent to the full forcing $\mathbb{B}$.

Lemma 3.2.1. *The random real forcing has the countable chain condition.*

Proof. Let $A \subseteq \mathbb{B}$ be an antichain in $\mathbb{B}$. Then if $p \in A$, there is some $n \in \mathbb{Z}$, and an $m \in \mathbb{N}$ such that $\mu(p \cap (n, n+1) > \frac{1}{m})$. So $A = \bigcup \{\{p \in A : \mu(p \cap (n, n+1)) > \frac{1}{m}\} : (n, m) \in \mathbb{Z} \times \mathbb{N}\}$, and for each (m, n), the set $\{p \in A : \mu(p \cap (n, n+1)) > \frac{1}{m}\}$ is finite (in fact it has cardinality less than m), and thus the union over the countable set $\mathbb{Z} \times \mathbb{N}$ is at most countable. $\square$

Theorem 3.2.2. *If G is a generic filter on $\mathbb{B}_C$ then $\bigcap\{p^{V[G]} :$ "$p \in V$ is an F-code" $\wedge\, p^V \in G\}$ contains a single element, and it is a real number of V[G].*

Proof. Since G is a filter, it has the finite intersection property. We can also see the sets $D_n := \{p \in \mathbb{B}_C : \text{diam}(p) < \frac{1}{n}\}$ are dense in $\mathbb{B}_C$. This of course means some of our sets are bounded, so by compactness the intersection $\bigcap\{p^{V[G]} :$ "$p \in V$ is an F-code" $\wedge\, p^V \in G\}$ is not empty in $V[G]$. Using the D_n's, we can see that this intersection contains a single point, let us call it a. $\square$

Theorem 3.2.3. *The generic filter G can be recovered from a. In particular $V[G] = V[a]$ and $a \notin V$.*

Proof. Let $H := \{p^V :$ "$p \in V$ is an F-code" $\wedge\, a \in p^{V[G]}\}$. We will now show that $G = H$.

It is almost trivial that $G \subseteq H$, because if $p \in V$ is an F-code such that $p^V \in G$, then by definition $a \in p^{V[G]}$, and thus $p^V \in H$. We now show inclusion in the other direction.

Let $p \in V$ be an F-code with $p^V \in H$, and extend the singleton $\{p^V\}$ to a maximal antichain A of $\mathbb{B}_C$ with the property that for every $p, q \in A$, $p \cap q = \emptyset$. We will show first that A is actually maximal with respect to all antichains of $\mathbb{B}_C$. Suppose that this is not the case, then let $r \in \mathbb{B}_C$ such that $A \cup \{r\}$ is an antichain in $\mathbb{B}_C$. Notice first that

$$\mu(r \backslash \bigcup A) = \mu(r) - \mu(r \cap \bigcup A) = \mu(r) - \sum_{p \in A} 0 = \mu(r)$$

So $r \backslash \bigcup$ A is a Borel set of positive measure, and thus contains a closed subset r' of positive measure. It is also clear that for any $s \in A$, $s \cap r' \subseteq s \cap (r \backslash \bigcup A) = \emptyset$. This of course is a contradiction to the maximality property of A, because $A \cup \{r\}$ is another antichain with pairwise disjoint elements. This means that A is already an antichain of $\mathbb{B}_C$ in the usual sense.

We know now that G must contain one element $q \in A$, so we assume that $p^V \neq q$. Since $q \in G$, it has an F-code $\hat{q} \in V$ such that $a \in \hat{q}^{V[G]}$, and thus $\hat{q}^{V[G]} \cap p^{V[G]} \neq \emptyset$. However as elements of A, we have $\hat{q}^V \cap p^V = \emptyset$. To show this is impossible, start by finding the n such that $a \in [n, n+1]^{V[G]}$. Now for each pair of rational numbers $\langle r, s\rangle \in p \cup \hat{q}$ notice that either $n < a < r$ or $s < a < n+1$, so $[n, n+1]^V \backslash (r, s)^V$ is either the set $[n, r]^V$ or $[s, n+1]^V$. If we take a finite collection of pairs, say for $m \in \omega$ we have the collection $\{\langle r_i, s_i\rangle : i \in m \wedge \langle r_i, s_i\rangle \in p \cup \hat{q}\}$, then the set $\bigcap\{[n, n+1]^V \backslash (r_i, s_i)^V : i \in m\}$ is nonempty. (In fact if we say $s_m := n$, and $r_m := n+1$ it is the closed interval $[\max\{s_i : i \leq m\}, \min\{r_i : i \leq m\}]^V$.) Thus $\{[n, n+1]^V \backslash (r, s)^V : \langle r, s\rangle \in p \cup \hat{q}\}$ is a collection of compact subsets of the real line (in V) that has the finite intersection property, and therefore has nonempty intersection. But

$$\bigcap\{[n, n+1]^V \backslash (r, s)^V : \langle r, s\rangle \in p \cup \hat{q}\} = (p^V \cap q) \cap [n, n+1] \subseteq (p^V \cap q) = \emptyset$$

which is a contradiction, so q and p must be one and the same.

We have proven that G can be recovered from a, so the extension is generated by the single real a. In particular $V[G] = V[a]$, and because it is clear that every condition has two extensions that are incompatible, we know $a \notin V$. $\square$

We want to show that the random real extension is different from the Cohen extension described in the previous section. To do this, we will show that every sequence of natural numbers in the extension is dominated by one in the ground model. To do that, we will borrow the notion of "truth value" from the Boolean algebra approach to forcing.

Definition. Let ϕ be a formula in the forcing language, then the *truth set* of ϕ (in the Solovay forcing) is defined by $|\phi| := \{\bigcup A : A \text{ is a maximal antichain of } \{p \in \mathbb{B} : p \Vdash \phi\}\}$.

Lemma 3.2.4. *Let ϕ be a formula, $r \in |\phi|$, and $q \in \mathbb{B}$. Then $q \leq r$ if and only if $q \Vdash \phi$. In particular $r \Vdash \phi$.*

Proof. Let A be a maximal antichain such that $\bigcup A = r$, and assume $q \Vdash \phi$. Suppose for contradiction that $q \not\leq r$. Then $\mu(q\backslash r) > 0$, so $q\backslash r \in \mathbb{B}$, and as we can see $q\backslash r \leq q$, so $q\backslash r \Vdash \phi$. This however contradicts the maximality of A, because $A \cup \{q\backslash r\}$ is an antichain of $\{p \in \mathbb{B} : p \Vdash \phi\}$.

Let us now suppose that $q \leq r$, but assume $q \nVdash \phi$. Then by the forcing theorem, there is some $s \leq q$ such that $s \Vdash \neg\phi$. This means however that given any $p \in \mathbb{B}$ such that $p \Vdash \phi$ we have $p \perp s$, in other words $\mu(s \cap p) = 0$. Thus $s \cap r = \bigcup(\{s \cap p : p \in A\}$ is a null set (because A is countable by the ccc), contradicting the fact that $s \subseteq q \subseteq r$ and all have positive measure. $\square$

Corollary 3.2.5. *Let ϕ be a formula and $q, r \in |\phi|$. Then q and r differ only by a null set. Formally, the symmetric difference $q\Delta r = (q\backslash r) \cup (r\backslash q)$ has measure zero. In particular $\mu(q) = \mu(r)$.*

Proof. Suppose for contradiction that $\mu(q\Delta r) > 0$. Then without loss of generality $\mu(q\backslash r) > 0$. But then $q\backslash r \in \mathbb{B}$, and is a stronger condition than q, and thus forces ϕ. But by the previous lemma $q\backslash r \subseteq r$, and it is clear that this is a contradiction. $\square$

In a sense, we see now that the truth set of ϕ is actually an equivalence class "maximal" of conditions that force ϕ. It will benefit us to talk about a representative of the class. From here on, we will let the notation $||\phi||$ mean to take any representative of the class $|\phi|$.

Lemma 3.2.6. *If $\dot{f} \in V$ is a name, and $p \in \mathbb{B}$ such that $p \Vdash \dot{f} : \omega \to \omega$, then for each $n \in \omega$ the set $F_n := \{||\dot{f}(n) = i|| \cap p : i \in \omega\}$ partitions p. (i.e. $\mu(\bigcup F_n) = \mu(p)$, and $\mu((||\dot{f}(n) = i|| \cap p) \cap (||\dot{f}(n) = j|| \cap p)) = 0$ if $i \neq j$.)*

Proof. Let $i, j \in \omega$ such that $i \neq j$, and let $p' := (||\dot{f}(n) = i|| \cap p) \cap (||\dot{f}(n) = j|| \cap p)$, then if $\mu(p') > 0$, $p' \in \mathbb{B}$, and has the properties that $p' \Vdash \dot{f} : \omega \to \omega$, $p' \Vdash \dot{f}(n) = i$, and $p' \Vdash \dot{f}(n) = j$. Since $i \neq j$, this is a contradiction.

Now let us suppose that $\mu(\bigcup F_n) \leq \mu(p)$, then $p\backslash \bigcup F_n \in \mathbb{B}$, and forces $\dot{f} : \omega \to \omega$, so there is some stronger condition $q \in \mathbb{B}$ and some $m \in \omega$ such that $q \Vdash \dot{f}(n) = m$. We now have by a previous lemma $q \leq ||\dot{f}(n) = m|| \leq \bigcup F_n$ but we chose $q \leq p\backslash \bigcup F_n$. This of course is impossible, so we must have $\mu(\bigcup F_n) = \mu(p)$. $\square$

Lemma 3.2.7. *In the random extension, every $f : \omega \to \omega$ is dominated by some $g \in V$.*

Proof. Let $f : \omega \to \omega$ be a function in $V[G]$, and let $\dot{f} \in V$ be a G-name for f. Then under each $p \in \mathbb{B}$ such that $p \Vdash \dot{f} : \omega \to \omega$ we will find a $p' \in \mathbb{B}$ and an $f_p \in V$, $f_p : \omega \to \omega$ such that $p' \Vdash \forall n \in \omega(\dot{f}(n) < f_p(n))$.

For each $n \in \omega$ find an $m_n \in \omega$ such that $\mu(p \backslash ||\dot{f}(n) < m_n||) < \frac{1}{4}(\frac{1}{2})^n \mu(p)$. This is possible because the set $\{||\dot{f}(n) = i|| \cap p : i \in \omega\}$ partitions p. Now we define $f_p(n) := m_n$, and let $p' := \bigcap\{||\dot{f}(n) < m_n|| : n \in \omega\}$. Notice that p' is a Borel set, and

$$\mu(p \backslash p') < \sum_{n \in \omega} (\frac{1}{4}(\frac{1}{2})^n \mu(p)) = \frac{1}{2}\mu(p)$$

so $\mu(p') \geq \frac{1}{2}\mu(p) > 0$. Thus p' is a forcing condition, and $p' \Vdash \forall n \in \omega(\dot{f}(n) < f_p(n))$. It is clear that the set $\{p' : p \in \mathbb{B} \wedge p \Vdash \dot{f} : \omega \to \omega\} \in V$ is dense in $\mathbb{B}$ "where we need it to be", so one such p' is in the generic filter, and $\dot{f}^G \in V[G]$ is dominated by some $f_p \in V$. □

Finally we end by classifying what exactly it means to be a *random real*.

Theorem 3.2.8. *A real number a is random over the ground model V if and only if for every null Borel set $q \in V$, and Borel code $p \in V$ such that $p^V = q$, we have $a \notin p^{V[a]}$. Since every null set is contained in a null G_δ set, we may assume that p is a G_δ-code.*

Proof. Suppose first that a is random over V, that is let $a = \bigcap G$ where G is a $\mathbb{B}$-generic filter over V. Let $p \in V$ be a G_δ-code for a null G_δ set in V. Then notice that $\{\mathbb{R}^V \backslash p^V\}$ is a maximal antichain in $\mathbb{B}$, so the single element $\mathbb{R}^V \backslash p^V$ is an element of G. This gives us however that $a \notin p^{V[a]}$.

Now let us assume that a is a real number such that $a \notin p^{V[a]}$ for all G_δ-codes $p \in V$ with p^V is a null set. Notice that this means $a \notin V$, because if it were, the singleton $\{a\}$ would be a null set. Then define a filter on $\mathbb{B}_C$ by $G := \{p^V :$ "$p \in V$ is an F-code" $\wedge\, a \in p^{V[a]}\}$. Then we claim that G is a generic filter, and $a \in \bigcap G$ is the unique random real generated by G.

1. G is not empty

   It is clear that $\mathbb{R}^V \in G$.

2. If $p < q$ and $p \in G$, then $q \in G$

   Since $p < q$ we have $p \backslash q$ is a null set. Then there are F-codes $\hat{p}$ and $\hat{q} \in V$, and a G_δ-code $\hat{r} \in V$ such that $\hat{p}^V = p$, $\hat{q}^V = q$, and $\hat{r}^V = p \backslash q$. From the assumption on a, we have that $a \notin \hat{r}^{V[G]}$. But $a \in \hat{p}^{V[G]}$, so a must be an element of $\hat{q}^{V[G]}$. Thus $q \in G$.

3. If $p, q \in G$ then there is an $r \in G$ such that $r \leq p, q$

   Since p, q are both elements of G, we have $a \in (\hat{p} \cap \hat{q})^{V[a]}$, for $\hat{p}$ and $\hat{q} \in V$ being G_δ-codes for p and q. From the assumption on a, this means $(p \cap q)$ can not be a null set in V, and thus has positive measure. Now take any closed set r contained in $(p \cap q)$ with $\mu(r) = \mu(p \cap q)$. Then $r \in G$ and $r \leq p, q$.

4. If A is a maximal antichain in $\mathbb{B}_C$ then $A \cap G \neq \emptyset$

   Since A is a maximal antichain, we know $\mathbb{R}^V \backslash \bigcup A$ is a null G_δ set. Thus $a \notin \mathbb{R}^{V[a]} \backslash \bigcup\{p^{V[G]} :$ "p is an F-code" $\wedge\, p^V \in A\}$, so for some F-code $p \in V$ with $p^V \in A$, $a \in p^{V[G]}$, which means $p^V \in A \cap G$.

We can see now that G is in fact a generic filter on $\mathbb{B}_C$ over V, and $a \in \bigcap\{p^V[G] : p^V \in G\}$, so a is a random real over V.

□

3.3 Sacks Reals

The next notion we shall look at is the Sacks forcing introduced by Gerald Sacks designed to produce a minimal model. We lose the countable chain condition with this model, but can show that cardinals are preserved for another reason. Our forcing conditions will be perfect trees, which we define now.

Definition. We say $p \subseteq 2^{<\omega}$ is a *tree* if for all $s \in p$ and for all $t \subseteq s$, $t \in p$. A tree $p \neq \emptyset$ is *perfect* if for all $s \in p$ there is a $t \supseteq s$ such that both $t^\frown 0 \in p$ and $t^\frown 1 \in p$.

Definition. Given a tree p, and an element $s \in p$, then we define the *restriction of p to s* as

$$p \upharpoonright s := \{t \in p : t \subseteq s \vee s \subseteq t\}$$

Corollary 3.3.1. *If p is perfect, and $s \in p$ then $p \upharpoonright s$ is perfect.*

Definition. Given a tree p, and $n \in \omega$ the n^{th} *level* of p is the set $p(n) := \{s \in p : |s| = n\}$

Lemma 3.3.2. *If p is a perfect tree, $p(n) \neq \emptyset$ for every $n \in \omega$.*

Proof. Suppose to the contrary that there was an $n \in \omega$ such that $p(n) = \emptyset$, and let n be minimal with this property. If $n = 0$, $p = \emptyset$, so p is not perfect. Otherwise, $p(n-1) \neq \emptyset$, so let $s \in p(n-1)$. Since p is perfect, there is some $t \in p$ such that $t \supseteq s$ and $t^\frown 0 \in p$, so $(t^\frown 0) \upharpoonright n \in p(n)$.

□

Forcing with Perfect Trees

Let $\mathbb{T}$ be the set of all perfect trees $p \subseteq 2^{<\omega}$, and say a tree $p \in \mathbb{T}$ is stronger than $q \in \mathbb{T}$ if and only if $p \subseteq q$. The maximal element of this forcing is the full tree $2^{<\omega}$.

It is not difficult to see that the Sacks forcing does not have the countable chain condition. What is clear is that $\mathrm{card}(\mathbb{T}) = 2^{\aleph_0}$, so in particular if $V \models CH$, we do have the $\aleph_2$-chain condition, and thus all cardinals greater than $\aleph_1$ are preserved. It is left to show that this forcing does not collapse $\aleph_1$, which we will do in the following lemmas.

Definition. In a tree p, we say $s \in p$ is a *branching point* if both $s^\frown 0 \in p$ and $s^\frown 1 \in p$. We say $s \in p$ is an n^{th} branching point if there are exactly n branching points contained in s. We say now $p \leq_n q$ if $p \subseteq q$ and if $s \in p$ is an n^{th} branching point of p, then it is an n^{th} branching point of q.

Corollary 3.3.3. *It is almost immediately clear that a tree is perfect if and only if it has 2^n many n^{th} branching points for each $n \in \omega$.*

Definition. A sequence of trees $\langle t_i : i \in \omega \rangle$ is called a *fusion sequence* if for all $i \in \omega$, $t_i \leq_i t_{i+1}$.

Lemma 3.3.4. *If $\langle t_i : i \in \omega \rangle$ is a fusion sequence of perfect trees, then $\bigcap\{t_i : i \in \omega\}$ is a perfect tree.*

Proof. If we let S_n be the set of n^{th} branching points of t_n, it is easy to see that S_n is the set of n^{th} branching points for all t_m with $m > n$, and thus of $\bigcap\{t_i : i \in \omega\}$. Since t_n is perfect, $|S_n| = 2^n$, which in turn tells us that $\bigcap\{t_i : i \in \omega\}$ has 2^n many n^{th} branching points, and is thus perfect. □

Lemma 3.3.5. *If A is a countable set of ordinals in $V[G]$ then there is a set $B \in V$ countable in V such that $A \subseteq B$.*

Proof. Let $f \in V[G]$ be a surjection $f : \omega \to A$. Then for each $p \in \mathbb{T}$ such that $p \Vdash \dot{f} : \omega \to \mathrm{ORD}$, we define $p' \leq p$ in the following way. First let $p_0 := p$. Now given p_n, we construct p_{n+1} as follows. Let S_n be the set of n^{th} branching points of p_n. Then for each $s \in S_n$ and $i \in 2$ find a condition $p_{s_i} \subseteq p \restriction (s^\frown i)$, and an ordinal a_{s_i} such that $p_{s_i} \Vdash \dot{f}(\check{n}) = a_{s_i}$. Then define $p_{n+1} := \bigcup\{p_{s_i} : s \in S_n \wedge i \in 2\}$.

Claim. *The sequence $\langle p_n : n \in \omega\rangle$ is a fusion sequence.*

Proof. It is clear that for each n, $p_n \subseteq p_{n+1}$. Now let s be an n^{th} branching point of p_n. Then $s^\frown i \in p_{s_i}$ for $i = 1, 2$, so both are elements of p_{n+1}, and thus s is a branching point of p_{n+1}. Now if $t \subsetneq s$ was also a branching point of p, then since p is perfect, there are n^{th} branching points t_0 and t_1 above $t^\frown 0$ and $t^\frown 1$ respectively. So both $t^\frown 0$ and $t^\frown 1$ are elements of p_{n+1}, and therefore s is an n^{th} branching point of s. □

Since this was a fusion sequence, it's intersection $p' := \bigcap\{p_n : n \in \omega\}$ is a condition of $\mathbb{T}$. Define $B_p := \{a_{s_i} : s \in S_n \wedge n \in \omega \wedge i \in \{0, 1\}\}$, which is obviously countable in V, and notice that $p' \Vdash \mathrm{Range}(A) \subseteq \check{B}_p$. Now the set $\{p' : p \in \mathbb{T} \wedge p \Vdash \dot{f} : \omega \to \mathrm{ORD}\}$ is dense in $\mathbb{T}$ "where we need it to be", so there is such a $p' \in G$, thus proving our theorem. □

Theorem 3.3.6. *If the ground model V satisfies CH, then the forcing $\mathbb{T}$ preserves all cardinals.*

Proof. It is clear that with CH $\mathrm{card}(\mathbb{T}) = \aleph_1$, and thus satisfies the $\aleph_2$-chain condition and preserves all cardinals greater than $\aleph_1$. Assume now that the forcing collapses $\aleph_1$. That is, ω_1^V is assumed to be countable in $V[G]$. Then by the previous lemma, in V we have ω_1^V is contained in a countable subset of V, which is a contradiction. □

Theorem 3.3.7. *If G is a generic filter on $\mathbb{T}$ then $f := \bigcup(\bigcap G)$ is a function $f : \omega \to 2$ and is thus a real number in $V[G]$.*

Proof. For each $n \in \omega$ define the set $D_n := \{p \in \mathbb{T} : |p(n)| = 1\}$. We claim that these sets are dense in $\mathbb{T}$. To show this, let $n \in \omega$, $p \in \mathbb{T}$, and take $s \in p(n)$. Then the tree $p \restriction s$ is perfect, and $(p \restriction s)(n) = \{s\}$, so it is an element of D_n.

Now let $s, t \in \bigcap G$ (notice that these are finite binary sequences and not trees), and let $n \in \mathrm{dom}(s) \cap \mathrm{dom}(t)$. Then choose a $p \in G \cap D_n$, and let p_1 be the unique element in $p(n)$. Since $s, t \in p$, and $n \in \mathrm{dom}(s) \cap \mathrm{dom}(p)$, we can see that $s \restriction (n+1) = p_1 = t \restriction (n+1)$. So s and t agree on any common domain, and f is a function $f : \mathrm{dom}(f) \to 2$.

Now we show that $\mathrm{dom}(f) = \omega$. For any $n \in \omega$ we can find a $p \in G \cap D_n$. Now let s be the unique element of $p(n+1)$. Let $q \in G$ and we will show that $s \in q$. Since $p, q \in G$, there is some $r \in G$ such that $r \subseteq p, q$. Since r is perfect, $r(n+1) \neq \emptyset$. Since $r(n+1) \subseteq p(n+1) = \{s\}$, $s \in r$, and thus $s \in q$. In particular $s \in \bigcap G$, so $n \in \mathrm{dom}(f)$.

□

Theorem 3.3.8. *The Sacks real described above is not a real in the ground model.*

Proof. Let $g \in V \cap 2^\omega$, and define the set $D_g := \{p \in \mathbb{T} : \exists n\ (f \restriction n \notin p)\}$. We show now that D_g is dense in $\mathbb{T}$. Let $p \in \mathbb{T}$, and assume that for all n we have $f \restriction n \in p$. Then let $s \in p$ be a branching point such that without loss of generality $s^\frown 0 \subseteq f$. Let $n \in \omega$ such that $s^\frown 0 = f \restriction n$. Then $p \restriction (s^\frown 1)$ is a perfect tree that does not contain $f \restriction n$, but extends p. We have now shown that D_g is dense, so there is some $p \in D_g \cap G$, and it is clear that $p \Vdash \check{g} \neq \bigcup\bigcap G$. □

Definition. If p is a tree, we define the *Cantor-Bendixon derivative* of p to be

$$p' := \{s \in p : \text{“}s \text{ is below a branching point of } p\text{”}\}$$

We let the $p^{(0)} := p$. If $p^{(\alpha)}$ is defined, we define $p^{(\alpha+1)} := (p^{(\alpha)})'$. If α is a limit ordinal, and $p^{(\beta)}$ is defined for every $\beta < \alpha$ we define $p^{(\alpha)} := \bigcap\{p^{(\beta)} : \beta < \alpha\}$.

Lemma 3.3.9. *Let p be a tree, then there exists a countable ordinal α such that $p^{(\alpha)} = p^{(\alpha+1)}$, and either $p^{(\alpha)}$ is empty or it is perfect.*

Proof. Notice that if $s \in p$ is removed at some step, so is the entire open set U_s (open in the space 2^ω). Since there are only countably many open sets in 2^ω, the decreasing sequence $\langle p^{(\alpha)} : \alpha \leq \omega_1\rangle$ has to stabilize after countably many steps. Let $\alpha < \omega_1$ be a countable ordinal such that $p^{(\alpha)} = p^{(\alpha+1)}$. Then if $p^{(\alpha)}$ is not empty, it is clear that every point in $p^{(\alpha)}$ lies below a branching point of $p^{(\alpha)}$, and thus the tree is perfect. □

Theorem 3.3.10. *If we let $H := \{p \in \mathbb{T} : \forall n \in \omega\ (f \restriction n \in p)\}$, then $G = H$. In particular G can be recovered from f, so $V[G] = V[f]$.*

Proof. If $p \in G$, it is clear that $f \restriction n \in p$ for all $n \in \omega$, so we have immediately that $G \subseteq H$. Now we show that H is a filter, and by **lemma 2.2.2** we are done.

1. H is not empty

   It is immediate that $2^\omega \in H$.

2. If $p \leq q$ and $p \in H$ then $q \in H$

   Let $n \in \omega$, then $f \restriction n \in p$ by the definition of H, and $f \restriction n \in q$ because $p \leq q$. Since this holds for all $n \in \omega$, $q \in H$.

3. If $p, q \in H$ then there is some $r \in H$ such that $r \leq p, q$

   Find a countable ordinal α such that $(p \cap q)^{(\alpha)} = (p \cap q)^{(\alpha+1)}$. We claim that $(p \cap q)^{(\alpha)}$ is perfect, and therefore a common extension of p and q. Suppose on the contrary that $(p \cap q)^{(\alpha)}$ is empty. Let $\beta < \alpha$ be minimal with the property $\exists n\ (f \restriction n \notin (p \cap q)^{(\beta)})$, and let $m \in \omega$ be the witness to this. It is clear that $\beta \neq 0$, because for every $n \in \omega$ we have $f \restriction n \in p \cap q)$. We also know that β is not a limit ordinal, because then $(p \cap q)^{(\beta)} := \bigcup\{(p \cap q)^{(\gamma)} : \gamma < \beta\}$, so if $f \restriction n \notin (p \cap q)^{(\beta)}$, then there is some $\gamma < \beta$ such that $f \restriction n \notin (p \cap q)^{(\gamma)}$. Notice however that $f = \bigcup\{s \in (p \cap q)^{(\beta-1)} : f \restriction m \subseteq s\}$, which is definable in V. We have however shown that $f \notin V$, so this cannot be true. Therefore $(p \cap q)^{(\alpha)}$ is a perfect tree and is also an extension of both p and q.

□

Theorem 3.3.11. *The Generic extension over $\mathbb{T}$ is minimal over the ground model V.*

Proof. We start by letting G be a generic filter on $\mathbb{T}$, taking $X \in V[G]$ as a subset of the ordinals such that $X \notin V$, and letting $\dot{X} \in V$ be a $\mathbb{T}$-name of X. For each $p \in G$ such that $p \Vdash \dot{X} \notin V$, we will find a $p' \leq p$ and a function $f_{p'} \in V[X]$ such that if $p' \in G$, then $f_{p'} = f = \bigcup\bigcap G$.

Start by letting p_0 be a condition such that $p \Vdash \dot{X} \notin V$. We will find p' by constructing a fusion sequence. For each $n \in \omega$ let S_n be the set of n^{th} branching points of p_n, and for each $s \in S_n$ find an ordinal γ_s such that $p_n \restriction s$ cannot decide $\check{\gamma}_n \in \dot{X}$. For each s, let $p_{s_0} \subseteq p \restriction (s^\frown 0)$, and $p_{s_1} \subseteq p \restriction (s^\frown 1)$ be conditions such that one forces $\check{\gamma}_s \in \dot{X}$, and the other forces $\check{\gamma}_s \notin \dot{X}$. Define

$p_{n+1} := \bigcup\{p_{s_i} : s \in S_n \wedge i \in 2\}$ and define $p' := \bigcap_{n\in\omega} p_n$.

We can see with the argument used in **lemma 3.3** that the sequence $\langle p_n : n \in \omega\rangle$ is a fusion sequence. We know then that p is a perfect tree, and thus a condition of $\mathbb{T}$.
Now for each $p \in \mathbb{T}$ define a function $f_p \in V[X]$ by:

$$\begin{aligned} f_p := \bigcup\{s^\frown i \in p : (s \text{ is a branching point of p}) \wedge (i \in 2)\wedge \\ (\forall\gamma \in \text{ORD } (\gamma \in X \wedge p \restriction (s^\frown i) \Vdash \check{\gamma} \in \dot{X})\vee \\ (\gamma \notin X \wedge p \restriction (s^\frown i) \Vdash \check{\gamma} \notin \dot{X})\vee \\ (p \restriction (s^\frown i) \nVdash \check{\gamma} \in \dot{X} \wedge p \restriction (s^\frown i) \nVdash \check{\gamma} \notin \dot{X}))\} \end{aligned}$$

Notice that if p' is the result of a fusion sequence defined above, and $p' \in G$, then $f_{p'} = f = \bigcup\bigcap G$. Since $\{p' : p \in \mathbb{T} \wedge p \Vdash \dot{X} \notin V\}$ is dense below every p that forces $\dot{X} \notin V$, we know there is such a $p' \in G$, and we are done. $\square$

4

Iterated Forcing

As Cohen states in [6], “Having shown how to adjoin one “generic” element... The obvious way [to violate the continuum hypothesis] is to adjoin sets of integers a_i where i ranges over all the ordinals less than $\aleph_2$”. We will look at methods of doing just that.

4.1 *The Two Step Iteration*

In our first method, we assume we have a forcing $P \in V$, and a forcing Q in the generic extension $V[G_P]$ (where G_P is a P-generic filter over V). We then want to construct a forcing $P * Q$ in the ground model that will give the same extension obtained by forcing first with P over V and then with Q over $V[G_P]$.

Definition. Suppose $(P, \leq_P, 1_P) \in M$ is a forcing, and further that $(\dot{Q}, \dot{\leq}_Q, \dot{1}_Q) \in M$ are names such that $1_P \Vdash [\dot{1}_Q \in Q \wedge$ “$(\dot{Q}, \dot{\leq}_Q, \dot{1}_Q)$ is a forcing with maximal element $\dot{1}_Q$”]. Then the *two-step iteration* $(P * \dot{Q}, \leq, 1)$ is defined by the following:

$$P * \dot{Q} := \{(p, \dot{q}) : p \in P \wedge \dot{q} \in \mathrm{Dom}(\dot{Q}) \wedge p \Vdash_P \dot{q} \in \dot{Q}\}$$

$$(p', \dot{q}') \leq (p, \dot{q}) \iff p' \leq_P p \wedge p' \Vdash_P \dot{q}' \dot{\leq}_Q \dot{q}$$

$$1 := (1_P, \dot{1}_Q)$$

Lemma 4.1.1. *The two step iteration described above is a forcing on M.*

Proof. It is clear from the definition that $P * Q \in M$, so we just need to show that the ordering $\leq$ is reflexive, transitive, and that 1 is the maximal element.

1. $\leq$ is reflexive

   If $(p, \dot{q}) \in P * \dot{Q}$ then clearly $p \leq_P p$ and since $1_p \Vdash$ “$\dot{Q}$ is a forcing”, $p \Vdash$ “$\dot{Q}$ is a forcing” $\wedge \dot{q} \in \dot{Q}$. So in particular $p \Vdash \dot{q} \dot{\leq}_Q \dot{q}$. So $(p, q) \leq (p, q)$.

2. $\leq$ is transitive

   Let $(p'', \dot{q}'') \leq (p', \dot{q}')$ and $(p', \dot{q}') \leq (p, \dot{q})$, with each pair an element of $P * \dot{Q}$. Then $p' \leq p$ and $p' \Vdash \dot{q}' \dot{\leq}_Q \dot{q}$ so because $p'' \leq p'$ and $p'' \Vdash \dot{q}'' \dot{\leq}_Q \dot{q}'$, we also get $p'' \leq p$ and $p'' \Vdash \dot{q}' \dot{\leq}_Q \dot{q} \wedge \dot{q}'' \dot{\leq}_Q \dot{q}'$. Since p'' also forces that $\dot{Q}$ is a forcing, we get $p'' \Vdash \dot{q}'' \dot{\leq}_Q \dot{q}$. Thus $(p'', \dot{q}'') \leq (p, \dot{q})$.

3. 1 is the maximal element of $P * \dot{Q}$

   Let $(p, \dot{q}) \in P * \dot{Q}$, then $1_P \leq p$, and $p \Vdash$ "$\dot{Q}$ is a forcing"$\wedge \dot{q} \in \dot{Q}$, we have $p \Vdash \dot{q} \dot{\leq}_Q \dot{1}_Q$.

So $P * \dot{Q}$ is a forcing in M. $\square$

Lemma 4.1.2. *If we have a two step iteration described above, and G is a P-generic filter over M, H is a $Q = \dot{Q}^G$-generic filter over $M[G]$, then $G * H := \{(p, \dot{q}) \in P * \dot{Q} : p \in G \wedge \dot{q}^G \in H\}$ is a $P * \dot{Q}$-generic filter over M.*

Proof.

1. $G * H$ is not empty

   True because $(1_P, \dot{1}_Q) \in G * H$.

2. If $(p, \dot{q}) \leq (p', \dot{q}')$ and $(p, \dot{q}) \in G * H$ then $(p', \dot{q}') \in G * H$

   Since $p \in G$ and $p \leq p'$, we have $p' \in G$. Now $p \Vdash \dot{q} \in \dot{Q} \wedge \dot{q}' \in \dot{Q} \wedge \dot{q}' \leq \dot{q}$, so in particular, since $p \in G$, $\dot{q}^G \leq \dot{q}'^G$ and both are elements of $\dot{Q}^G$. Since H is a filter and $\dot{q}^G \in H$, so is $\dot{q}'^G$. The result is $(p', \dot{q}') \in G * H$.

3. If $(p, \dot{q}), (p', \dot{q}') \in G*H$ then there is some $(p'', \dot{q}'') \in G*H$ such that $(p'', \dot{q}'') \leq (p, \dot{q}), (p', \dot{q}')$

   Since $\dot{q}^G, \dot{q}'^G \in H$, and H is a filter, find some $q'' \in H$ such that $q'' \leq \dot{q}^G, \dot{q}'^G$. Now find a name $\dot{q}'' \in \mathrm{Dom}(\dot{Q})$ for q''. Similarly, find a $p'' \leq p, p'$, but let us choose it in a way that p'' also forces $\dot{q}'' \in \dot{Q}$, $\dot{q}'' \leq \dot{q}$, and $\dot{q}'' \leq \dot{q}'$. Then $(p'', \dot{q}'') \in G * H$ and is stronger than both $(p, \dot{q})$ and $(p', \dot{q}')$ as desired.

4. If $D \in M$ is dense in $P * Q$ then $G * H \cap D \neq \emptyset$

   First, define the set $D_Q := \{\dot{q}^G : \dot{q} \in \mathrm{Dom}(\dot{Q}) \wedge \exists p \in G(p, \dot{q}) \in D\}$. We will show that this set is dense in Q over $M[G]$. Let $q \in Q$, then there is some P-name $\dot{q} \in \mathrm{Dom}(\dot{Q})$ such that $\dot{q}^G = q$, and we can find a $p \in G$ such that $p \Vdash \dot{q} \in \dot{Q}$. Then it is clear that $(p, \dot{q}) \in P * \dot{Q}$.

   Now define the set $D_{(p,\dot{q})} := \{r \in P : r \leq p \wedge \exists \dot{s} \in \mathrm{Dom}(\dot{Q})[(r, \dot{s}) \in D \wedge p \Vdash \dot{s} \leq \dot{q}]\}$, and we claim this set is dense below $p \in P$. To show this, let $p' \leq p$, then $p' \Vdash \dot{q} \in \dot{Q}$, so $(p', \dot{q}) \in P * \dot{Q}$. By the density of D there is some $(r, \dot{s}) \leq (p'\dot{q})$ such that $(r, \dot{s}) \in D$. We have then by definition $r \leq p$ and $r \Vdash \dot{s} \leq \dot{q}$, so $r \in D_{(p,\dot{q})}$ and $r \leq p'$, proving that $D_{(p,\dot{q})}$ is dense under p.

   By the genericity of G, we now know that there is some $p' \leq p$ such that $p' \in D_{(p,\dot{q})} \cap G$. So for some $\dot{q}'$, $(p', \dot{q}') \in D$ and $p' \Vdash \dot{q}' \leq \dot{q}$. Since $p' \in G$, $\dot{q}'^G \leq \dot{q}^G = q$ and $\dot{q}'^G \in D_Q$, proving the density of D_Q that we wanted in Q.

   Now since D_Q is dense in Q over $M[G]$, there is some $\dot{q}^G \in D_Q \cap H$. Since this $\dot{q}^G \in D_Q$, there is some $p \in G$ such that $(p, \dot{q}) \in D$. Thus $(p, \dot{q}) \in G * H \cap D$, proving that $G * H$ is a $P * \dot{Q}$-generic filter over M.

$\square$

Theorem 4.1.3. *Forcing with the two step iteration $P * \dot{Q}$ is equivalent to forcing first with P and then with the interpretation of $\dot{Q}$ in the P-generic extension.*

Proof. Let G be a $P*\dot{Q}$-generic filter over V. Define then $G_1 := \{p \in P : \exists \dot{q} \in \mathrm{dom}(\dot{Q})\ (p, \dot{q}) \in G\}$ and $G_2 := \{\dot{q}^{G_1} : \exists p \in P\ (p, \dot{q}) \in G\}$. We claim then that G_1 is P-generic over V, G_2 is $\dot{Q}^{G_1}$-generic over $V[G_1]$, and $G = G_1 * G_2$.

1. G_1 is not empty

   This follows from G is not empty.

2. If $p \leq p'$ and $p \in G_1$ then $p' \in G_1$

   Since $p \in G_1$, there is some $\dot{q}$ such that $(p, \dot{q}) \in G$. Then $(p, \dot{q}) \leq (p', \dot{q})$, so $(p', \dot{q}) \in G$ and thus $p' \in G_1$.

3. If $p, p' \in G_1$ then there is some $p'' \in G_1$ such that $p'' \leq p, p'$

   Similarly take $(p, \dot{q}), (p', \dot{q}') \in G$, then there is a $(p'', \dot{q}'') \in G$ that extends both. Then $p'' \in G_1$ and $p'' \leq p, p'$.

4. If $D \in M$ is dense in P then $G_1 \cap D \neq \emptyset$

   Define $D * \dot{Q} := \{(p, \dot{q}) : p \in D \wedge \dot{q} \in \mathrm{dom}(\dot{Q})\}$. This set is dense in $P * \dot{Q}$, so there is some $(p, \dot{q}) \in (D * \dot{Q}) \cap G$. Clearly $p \in D \cap G_1$, proving our claim.

The proof for G_2 is similar. □

4.2 Finite and Countable Support Iterations

We would like to be able to add a large number of generic elements to our ground model. At the same time, we would like to preserve some of the nicer properties of forcing notions. Here we introduce two methods of iterated forcing.

Definition. A sequence $\langle P_\alpha, \leq_{P_\alpha}, 1_{P_\alpha}; \dot{Q}_\beta, \dot{\leq}_{Q_\beta} : \beta < \gamma, \alpha \leq \gamma \rangle$ is a *(finite) countable support iteration* if the following hold:

1. The $(P_\alpha, \leq_{P_\alpha}, 1_{P_\alpha})$ are forcing notions.

2. If $x \in P_\alpha$ then it is a function $x : \alpha \to V$. The support of x is defined by

$$\mathrm{supp}(x) := \{\beta < \alpha : x(\beta) \neq \emptyset\}$$

3. If $\alpha \leq \gamma$ is a limit ordinal, then:

   (a) $P_\alpha = \{p : \alpha \to V : \forall \beta < \alpha (p \restriction \beta \in P_\beta) \wedge$ supp(p) is (finite) at most countable$\}$

   (b) $p \leq_{P_\alpha} q$ if and only if $\forall \beta < \alpha\ (p \restriction \beta \leq_{P_\beta} q \restriction \beta)$

   (c) $1_{P_\alpha}(\beta) = \emptyset$ for all $\beta < \alpha$

   (Notice that by the above definition P_0 is the trivial forcing: $\langle \{\emptyset\}, \langle \emptyset, \emptyset \rangle, \emptyset \rangle$.)

4. For every $\alpha < \gamma$ $(1_{P_\alpha} \Vdash_{P_\alpha} (\dot{Q}_\alpha, \dot{\leq}_{Q_\alpha}, \emptyset)$ is a forcing)

5. If $\alpha < \gamma$ then

   (a) $P_{\alpha+1} = \{p : (\alpha+1) \to V : p \restriction \alpha \in P_\alpha \wedge p(\alpha) \in \mathrm{dom}(\dot{Q}_\alpha) \wedge (p \restriction \alpha) \Vdash_{P_\alpha} p(\alpha) \in \dot{Q}_\alpha\}$

   (b) $p \leq_{\alpha+1} q$ if and only if $p \restriction \alpha \leq_\alpha q \restriction \alpha \wedge (p \restriction \alpha) \Vdash_{P_\alpha} p(\alpha) \leq_\alpha q(\alpha)$

(c) $1_{P_{\alpha+1}}(\beta) = \emptyset$ for all $\beta < \alpha + 1$

We may also abuse notation in the normal way, and write simply $\langle P_\alpha, \dot{Q}_\alpha : \alpha \leq \gamma \rangle$.

Remark. Notice first that if $\alpha \leq \beta$ there is a natural inclusion of $P_\alpha \hookrightarrow P_\beta$, namely $p \mapsto p'$, where $p'(\xi) := \begin{cases} p(\xi) & \text{if } \xi < \alpha \\ \emptyset & \text{if } \alpha \leq \xi \leq \gamma \end{cases}$. Through this definition, we can think of any P_α-name as a P_β-name, and any P_α-sentence as a P_β-sentence.

When we are working with ccc forcing notions, the "right" type of iteration to use is finite support, because the ccc property will be preserved. However, we will soon introduce the idea of properness, which is a generalization of the ccc. When working with proper forcing notions, we will be able to relax our iteration method to countable supports, while still preserving this nice property.

Remark. The two step iteration is a finite or countable support iteration where $\gamma = 2$ by the following. Suppose $P \in V$ is a forcing, and $\dot{Q}$ is a P-name such that $1_P \Vdash \dot{Q}$ is a forcing. Then let P_0 be the trivial forcing, let $\dot{Q}_0 = \check{P}$, where $\check{P}$ is the canonical P_0-name for P, let $P_1 := \{f : 1 \to V | f(0) \in \mathrm{dom}(\dot{Q}_0)\}$. Now it is clear that P_1 is isomorphic as a forcing to P, so we can take $\dot{Q}$ as a P_1 name, and define $\dot{Q}_1 := \dot{Q}$. Finally, let
$P_2 := \{f : 2 \to V | f(0) \in \mathrm{dom}(\dot{Q}_0) \wedge f(1) \in \dot{Q}_1\}$. Then $\langle P_i, \dot{Q}_i : i \leq 2 \rangle$ is a countable support iteration, and is isomorphic to $P * \dot{Q}$.

Lemma 4.2.1. ***(Properties of Countable Support Iterations)***
If $\langle P_\alpha, \dot{Q}_\alpha : \alpha \leq \gamma \rangle$ is a countable support iteration, and $\alpha \leq \beta \leq \gamma$ then the following hold.

1. $\forall p \in P_\beta \ ((p \restriction \alpha) \in P_\alpha)$
2. $\forall p \in P_\beta \ ((p \restriction \alpha) \leq_{P_\beta} p)$
3. $\forall p, q \in P_\beta \ (p \leq_{P_\beta} q \to (p \restriction \alpha) \leq_{P_\alpha} (q \restriction \alpha))$
4. $\forall p, q \in P_\beta \ ((p \restriction \alpha) \perp_{P_\alpha} (q \restriction \alpha) \to p \perp_{P_\beta} q)$
5. $\forall p \in P_\alpha \forall q \in P_\beta \ (p \leq_{P_\alpha} (q \restriction \alpha) \leftrightarrow p \leq_{P_\beta} q)$
6. *If G_β is a P_β generic filter, then $G_\alpha := \{p \restriction \alpha : p \in G_\beta\}$ is a P_α-generic filter*

5

Proper Forcing

As we saw earlier, the countable chain condition is a very nice property for a forcing notion to have, because with it, the forcing will preserve all cardinals and cofinalities. Unfortunately, when iterating, we have to use finite supports if we hope to preserve this property. Fortunately however, there is a more general property, namely properness, that we shall introduce in this chapter. Properness was introduced by Saharon Shelah, and was designed specifically to not collapse $\aleph_1$. If we are lucky enough to also have the $\aleph_2$-chain condition, we can see that all cardinals are preserved under this type of forcing. In [13], Shelah uses stationary sets to define his forcing notion. We however will be starting from a different definition (proved by Shelah on page 102 of [13] to be equivalent) that uses elementary substructures of H_λ. While everything here can be found in Shelah's book, we will be following [1] more closely.

5.1 Generic Conditions and Proper Forcing

Definition. Let $P \in V$ be a forcing, let λ be an ordinal, and let $M \prec H_\lambda$ such that $P \in M$. Then $q \in P$ is (M,P)-*generic* if for every dense subset $D \subseteq P$ such that $D \in M$, $D \cap M$ is predense below q.

Lemma 5.1.1. *A condition q is (M,P)-generic if and only if for every $D \in M$ that is dense in P, there is a P-name $\dot{p} \in V$ such that $q \Vdash_P \dot{p} \in \check{M} \cap \check{D} \cap \dot{G}$.*

Proof. Suppose first that q is (M,P)-generic, let G be an arbitrary P-generic filter over V such that $q \in G$, and let $D \in M$ be a dense subset of P. Since $D \cap M$ is predense below q and $q \in G$, we have $D \cap M \cap G \neq \emptyset$, and thus there is some p in this intersection. Since G was chosen arbitrarily to contain q, by the maximality principal, there is some $\dot{p} \in V$ such that $q \Vdash_P \dot{p} \in \check{M} \cap \check{D} \cap \dot{G}$.
For the other direction, let $q \in P$, and suppose that the right hand statement is true, then let $D \in M$ be a dense subset of P, and let $\dot{p}$ be the condition such that $q \Vdash_P \dot{p} \in \check{M} \cap \check{D} \cap \dot{G}$, let $q' \leq q$, and let G be a P-generic filter over V such that $q' \in G$. Then by the assumption, $\dot{p}^G \in M \cap D \cap G$. Since G is a filter, there is some $q'' \in G$ that extends both $\dot{p}^G$ and q'. Thus by definition $D \cap M$ is predense below q. □

Lemma 5.1.2. *A condition q is (M,P)-generic if and only if $q \Vdash M[\dot{G}] \cap Ord = M \cap Ord$.*

Proof. Please see [13] chapter III, lemma 2.6. □

Definition. A forcing is called *proper* if for every $\lambda > 2^{\mathrm{card}(P)}$ and every countable $M \prec H_\lambda$ with $P \in M$, every condition in $P \cap M$ has an (M,P)-generic extension.

5.2 Two Step Iteration of Proper Forcing Notions

Our first goal is to show that properness is preserved in two-step iterations of proper forcing notions. We are actually going to prove a much stronger condition below, but first we need a lemma help us find generic conditions in our two-step iteration.

Lemma 5.2.1. *Let $P * \dot{Q}$ be a two step iteration of forcing notions, and let λ be sufficiently large with $M \prec H_\lambda$, and $P * \dot{Q} \in M$. Further let $p \in P$ be an (M, P)-generic and $\dot{q} \in V$ be a P-name such that $p \Vdash$ "$\dot{q}$ is $(M[\dot{G}_p], \dot{Q})$-generic". Then $(p, \dot{q})$ is $(M, P * \dot{Q})$-generic.*

Proof. Suppose p and q are as in the statement. Then let G be a $(P * \dot{Q})$-generic filter over V, let $x \in M[G] \cap \mathrm{Ord} = M[G_1][G_1] \cap \mathrm{Ord}$, and let $\dot{x} \in M[G_1]$ be a $\dot{Q}^{G_1}$ name such that $\dot{x}^{G_2} = x$. Since $p \in G_1$, $\dot{q}^{G_1} \Vdash M[\check{G}_1][\dot{G}_2] \cap \mathrm{Ord} = M[\check{G}_1] \cap \mathrm{Ord}$, so $x = \dot{x}^{G_2} \in M[G_1] \cap \mathrm{Ord}$. Thus there is a P-name $\ddot{x} \in M$ such that $\ddot{x}^{G_1} = \dot{x}^{G_2} = x \in M[G_1] \cap \mathrm{Ord}$. Since p is (M, P)-generic, we have $x = \ddot{x}^{G_1} \in M \cap \mathrm{Ord}$. Since x was arbitrary, we have $M[G] \cap \mathrm{Ord} = M \cap \mathrm{Ord}$. Since the only restriction we had on G was that $(p, \dot{q}) \in G$, we see now that $(p, \dot{q}) \Vdash M[\dot{G}] \cap \mathrm{Ord} = M \cap \mathrm{Ord}$, so $(p, \dot{q})$ is $(P * \dot{Q})$-generic. □

Lemma 5.2.2. ***(The Two-Step Properness Extension Lemma)***
Let P be a proper forcing over V, $\dot{G}_P$ be the canonical name for a P-generic filter over V, and $\dot{Q}$ be a P-name such that

$$1_P \Vdash_P \text{"} \dot{Q} \textit{ is a proper forcing over } V[\dot{G}_P] \text{"}$$

Let λ be sufficiently large, and let $M \prec H_\lambda$ be countable such that $P * \dot{Q} \in M$. Assume that $p \in P$ is an (M, P)-generic condition, and $\dot{r} \in V$ is a P-name such that

$$p \Vdash_P \text{"}\dot{r} \in M \cap (P * \dot{Q}) \textit{ and } \pi_1(\dot{r}) \in \dot{G}_P\text{"}$$

Then there is some P-name, $\dot{q} \in V$ such that $(p, \dot{q})$ is $(M, P * \dot{Q})$-generic, and

$$(p, \dot{q}) \Vdash_{(P*\dot{Q})} \dot{r} \in \dot{G}$$

(Where $\dot{G}$ is the canonical $P * \dot{Q}$ name for a $(P * \dot{Q})$-generic filter over V).

Proof. Let G_P be some (V, P)-generic filter such that $p \in G_P$. Then $\dot{r}^{G_P} \in M \cap (P * \dot{Q})$, and thus has the form $\dot{r}^{G_P} = (r_0, \dot{r_1})$ and $\pi_1(\dot{r}^{G_P}) = r_0 \in \dot{Q}^{G_P}$. Now $\dot{r_1}$ is a P-name for a condition in $\dot{Q}$, and can thus be interpreted with G_P, giving us $\dot{r_1}^{G_P} \in \dot{Q}^{G_P}$. Since $\dot{Q}^{G_P}$ is proper over $V[G_P]$, and $\dot{r_1}^{G_P} \in \dot{Q}^{G_P} \cap M[G_P]$ ($\dot{r}_1 \in M$ because $(r_0, \dot{r_1}) \in M$), there is a $(M[G_P], \dot{Q}^{G_P})$-generic extension $q \in \dot{Q}^{G_P}$ of $\dot{r}_1^{G_P}$. Since G_P was an arbitrary (V, P)-generic filter containing p, by the maximality principal we can find a P-name $\dot{q}$ of q such that:

$$p \Vdash_P \text{"}\dot{q} \text{ is an } (M[\dot{G}_P], \dot{Q})\text{-generic extension of } \pi_2(\dot{r})\text{"}$$

By **lemma 5.2.1**, we already have that $(p, \dot{q})$ is $(M, P * \dot{Q})$-generic, so it is left to show that

$$(p, \dot{q}) \Vdash_{(P*\dot{Q})} \dot{r} \in \dot{G}$$

First remember that $\dot{r}$ is a P-name, and $p \Vdash_P \dot{r} \in (P * \dot{Q})$, so in particular, if we think of $\dot{r}$ as a $P * \dot{Q}$ name, we have $(p, \dot{q}) \Vdash_{(P*\dot{Q})} \dot{r} \in (P * \dot{Q})$. Now suppose $(p', \dot{q}')$ is some extension of $(p, \dot{q})$ that decides $\dot{r}$ to be some $(r_0, \dot{r_1}) \in P * \dot{Q}$, that is $(p', \dot{q}') \Vdash_{(P*\dot{Q})} \dot{r} = (r_0, \dot{r_1})\check{}$. Then $p' \Vdash_P \pi_1(\dot{r}) \in \dot{G}_P$, and thus since P is separative $p' \leq r_0$. Since $p \Vdash_P$ "$\dot{q}$ extends $\pi_2(\dot{r})$", and we have $p' \Vdash_P$ "$\dot{q}'$ extends $\dot{q}$ and $\dot{q}$ extends $\dot{r_1}$", so $(p', \dot{q}') \leq (r_0, \dot{r_1})$ and thus $(p', \dot{q}') \Vdash_{(P*\dot{Q})} \dot{r} \in \dot{G}$. Since this is true for any extension of $(p, \dot{q})$ that identifies $\dot{r}$, we have the desired result that $(p, \dot{q}) \Vdash_{(P*\dot{Q})} \dot{r} \in \dot{G}$. □

It follows almost directly that properness is preserved in a two step iteration.

Theorem 5.2.3. *If P is a proper forcing over V, $\dot{G}_P$ is the canonical name for a P-generic filter over V, and $\dot{Q}$ is a P-name such that*

$$1_P \Vdash_P \text{ “}\dot{Q} \textit{ is a proper forcing over } V[\dot{G}_P]\text{”}$$

Then $P * \dot{Q}$ is a proper forcing over V.

Proof. Let λ be sufficiently large, and $M \prec H_\lambda$ be a countable submodel such that $P * \dot{Q} \in M$, and let $(p, \dot{q}) \in M \cap P * \dot{Q}$. Then $p \in M \cap P$, and P is proper over V, so there is some (M, P)-generic extension $p' \in P$ of p. Now let $\dot{r}$ be the canonical P-name for the condition $(p, \dot{q})$. It is clear that

$$p' \Vdash_P \text{“}\dot{r} \in M \cap P * \dot{Q} \text{ and } \pi_1(\dot{r}) \in \dot{G}_P\text{”}$$

So by the previous lemma, there is some P-name $\dot{q}'$ such that $(p', \dot{q}')$ is $(M, P * \dot{Q})$-generic, and $(p', \dot{q}') \Vdash_{(P*\dot{Q})} \dot{r} \in \dot{G}$. Since $\dot{r}$ was the canonical P-name of $(p, \dot{q})$, we actually have $(p', \dot{q}') \Vdash_{(P*\dot{Q})} (p, \dot{q})\check{} \in \dot{G}$. Since $P * \dot{Q}$ is separative, $(p', \dot{q}') \leq (p, \dot{q})$, and thus $P * \dot{Q}$ is proper over V. □

5.3 Countable Support Iteration of Proper Forcing Notions

We now show that properness is preserved in countable support iterations. Again, we prove first a much stronger condition similar to the one from the two-step version.

Definition. A countable support iteration $\langle P_\alpha, \dot{Q}_\alpha : \alpha \leq \gamma \rangle$ is an *iteration of proper forcing notions* if for every $\alpha < \gamma$, $1_{P_\alpha} \Vdash_{P_\alpha}$ “$\dot{Q}_\alpha$ is a proper forcing”.

Lemma 5.3.1. ***(The Properness Extension Lemma)***
Let $\langle P_\alpha, \dot{Q}_\alpha : \alpha \leq \gamma \rangle$ be a countable support iteration of proper forcing notions, and for each α, let $\dot{G}_\alpha$ be the canonical P_α-name for a P_α-generic filter over V. Let λ be sufficiently large, and $M \prec H_\lambda$ be countable with $\gamma, P_\gamma, \langle P_\alpha, \dot{Q}_\alpha : \alpha \leq \gamma \rangle \in M$. Let $\gamma_0 \in \gamma \cap M$, and assume that $p_{\gamma_0} \in P_{\gamma_0}$ is an (M, P_{γ_0})-generic condition and $\dot{r} \in V$ is a P_{γ_0}-name such that:

$$p_{\gamma_0} \Vdash_{P_{\gamma_0}} \text{“}\dot{r} \in M \cap P_\gamma \textit{ and } \dot{r} \restriction \gamma_0 \in \dot{G}_{\gamma_0}\text{”}$$

Then there is some (M, P_γ)-generic condition p such that $p \restriction \gamma_0 = p_{\gamma_0}$, and

$$p \Vdash_{P_\gamma} \dot{r} \in \dot{G}_\gamma$$

Proof. We prove this lemma by induction on γ. Assume first that γ is a successor. If $\gamma_0 + 1 = \gamma$, then we have $P_\gamma = P_{\gamma_0} * \dot{Q}_{\gamma_0}$, and this case is handled by **lemma 5.2.2**.

Assume then that $\gamma_0 < \gamma' < \gamma' + 1 = \gamma$. By induction, the statement of the lemma holds for γ'. So since

$$p_{\gamma_0} \Vdash_{P_{\gamma_0}} \text{“}\dot{r} \in M \cap P_\gamma \text{ and } \dot{r} \restriction \gamma_0 \in \dot{G}_{\gamma_0}\text{”}$$

We have by definition

$$p_{\gamma_0} \Vdash_{P_{\gamma_0}} \text{“}\dot{r} \restriction \gamma' \in M \cap P_{\gamma'} \text{ and } (\dot{r} \restriction \gamma') \restriction \gamma_0 \in \dot{G}_{\gamma_0}\text{”}$$

Then by the statement of the lemma applied to γ' and $\dot{r} \restriction \gamma'$, there is some $(M, P_{\gamma'})$-generic condition p' such that $p' \restriction \gamma_0 = p_{\gamma_0}$ and

$$p' \Vdash_{P_{\gamma'}} \text{“}\dot{r} \in M \cap P_\gamma \text{ and } \dot{r} \restriction \gamma' \in \dot{G}_{\gamma'}\text{”}$$

We have now reduced this case to the previous one where $\gamma_0 + 1 = \gamma$.

So we are left with the case that γ is a limit ordinal. In this case, we start by choosing an increasing sequence $\langle \gamma_i : i \in \omega \rangle \in V$ that is cofinal in $\gamma \cap M$, where γ_0 is the same γ_0 given in the statement of the lemma. This is of course possible, since M is countable. Let us enumerate all of the dense sets of P_γ that are in M, say as $\langle D_i : i \in \omega \rangle$. We will now define by induction a sequence of P_{γ_n}-names $\dot{r}_n$, and a sequence of conditions p_n with each $p_n \in P_{\gamma_n}$ such that for each n, $p_{n+1} \restriction n = p_n$ and

$$
\begin{aligned}
p_n \Vdash_{P_{\gamma_n}} \text{“}\dot{r}_n &\in P_\gamma \cap M && (5.1)\\
\dot{r}_n \restriction \gamma_n &\in \dot{G}_{\gamma_n} && (5.2)\\
\dot{r}_{n-1} &\leq \dot{r}_n \text{ (for } n > 0) && (5.3)\\
\dot{r}_n &\in D_{n-1} \text{ (for } n > 0)\text{”} && (5.4)
\end{aligned}
$$

We start by letting $\dot{r}_0$ and p_0 be the given conditions, $\dot{r}$ and p_{γ_0} respectively. Then we inductively define a P_{γ_n}-name $\dot{r}_{n+1}$ and an (M, P_{γ_n})-generic condition p_n for each $n \in \omega$ as follows.

First from $\dot{r}_n$ and p_n we construct $\dot{r}_{n+1}$. Assume G_n is a P_{γ_n}-generic filter over V such that $p_n \in G_n$, and define $r_n := \dot{r}_n^{G_n}$. Notice now that (5.1)-(5.4) hold for r_n in $V[G]$. Now in V define

$$D'_n := \{p \restriction \gamma_n : p \in D_n \wedge [(p \leq_{P_\gamma} r_n) \vee (p \restriction \gamma_n \perp r_n \restriction \gamma_n)]\}$$

Notice that $D'_n \in M$, as D_n, γ_n, and $r_n \in M$. Now we show that D'_n is a dense subset of P_{γ_n}. To do this, let $q \in P_{\gamma_n}$.

1. If $q \perp (r_n \restriction \gamma_n)$ in P_{γ_n}, then as an elements of P_γ, $q \perp r_n$. By the density of D_n, we can choose some $p \in D_n$ with $p \leq_{P_\gamma} q$. We have $(p \restriction \gamma_n) \leq_{P_{\gamma_n}} q$, so $(p \restriction \gamma_n) \perp (r_n \restriction \gamma_n)$, $(p \restriction \gamma_n) \in D'_n$, with $(p \restriction \gamma_n) \leq_{P_{\gamma_n}} q$.
2. If $q \parallel (r_n \restriction \gamma_n)$ in P_{γ_n}, then as elements of P_γ, $q \parallel r_n$. So we can find a condition in P_γ stronger than both q and r_n, and by the density of D_n, we can find a $p \in D$ stronger than this condition. Since $p \leq_{P_\gamma} r_n$, $(p \restriction \gamma_n) \in D'_n$ and we already know $(p \restriction \gamma_n) \leq q$.

Now p_n is (M, P_{γ_n})-generic, and $D'_n \in M$ is a dense subset of P_{γ_n}, so $D'_n \cap M$ is predense below p_n. Since $p_n \in G_n$, and G_n is a generic filter, we know that $G_n \cap D'_n \cap M \neq \emptyset$. So take some x in this intersection, and notice by the definition of D' that there is some $r \in D_n$ such that $r \restriction \gamma_n = x$, in particular, $H_\lambda \models \exists r \in D_n(r \restriction \gamma_n = x)$. Since D_n, γ_n, and x are all elements of M, by the elementarity of M this existential statement holds in M as well, so we can find some $r_{n+1} \in M$ such that $r_{n+1} \in D_n$ and $(r_{n+1} \restriction \gamma_n) = x \in G_n$. Since $r_{n+1} \restriction \gamma_n$ and $r_n \restriction \gamma_n$ are both elements of G_n, they are compatible. Thus by the definition of D'_n we actually know that $r_{n+1} \leq_{P_\gamma} r_n$.

Since G_n was an arbitrary P_{γ_n}-generic filter containing p_n, by the maximality principal, we can find a P_{γ_n}-name $\dot{r}_{n+1}$ for r_{n+1} such that

$$
\begin{aligned}
p_n \Vdash_{P_{\gamma_n}} \text{“}\dot{r}_{n+1} &\in \check{P}_\gamma \cap \check{M}\\
\dot{r}_{n+1} \restriction \gamma_n &\in \dot{G}_{\gamma_n}\\
\dot{r}_{n+1} &\leq_{P_\gamma} \dot{r}_n\\
\dot{r}_{n+1} &\in \check{D}_n
\end{aligned}
$$

By induction, we can now apply the statement of the lemma to $P_{\gamma_{n+1}}$, γ_n, p_n, and $\dot{r}_{n+1}$ to find an $(M, P_{\gamma_{n+1}})$-generic condition p_{n+1} such that $p_{n+1} \restriction n = p_n$, (and thus forces the three statements above about $\dot{r}_{n+1}$), and forces the stronger statement $\dot{r}_{n+1} \restriction \gamma_{n+1} \in \dot{G}_{\gamma_{n+1}}$.

Since for every $m \leq n$ we have $p_n \restriction m = p_m$, the union $\bigcup_{n\in\omega} p_n$ is a function with countable support, and in particular is a forcing condition in P_γ. We now define $p := \bigcup_{n\in\omega} p_n$, and claim this is the (M, P_γ)-generic condition we are looking for. To prove this, notice first that it follows from (5.1)-(5.4) that

$$p \Vdash_{P_\gamma} \text{“}\dot{r}_n \in \check{P}_\gamma \cap \check{M} \tag{5.5}$$

$$\dot{r}_n \restriction \gamma_n \in \dot{G}_{\gamma_n} \tag{5.6}$$

$$\dot{r}_{n+1} \leq_{P_\gamma} \dot{r}_n \tag{5.7}$$

$$\dot{r}_{n+1} \in D_n\text{”} \tag{5.8}$$

The condition p is (M, P_γ)-generic because if D is a dense subset of P_γ in M, then there is some n such that $D = D_n$. Then with (5.8) and **lemma 5.1.1** p is (M, P_γ)-generic.

It follows from (5.7) that if $n \leq m$ then:

$$p \Vdash \dot{r}_m \leq_{P_\gamma} \dot{r}_n$$

Together with (5.6) this yields for $n \leq m$:

$$p \Vdash \dot{r}_n \restriction \gamma_m \in \dot{G}_{\gamma_m}$$

Remembering that $\dot{r}_0 = \dot{r}$, in particular we have for all $m \in \omega$

$$p \Vdash \dot{r} \restriction \gamma_m \in \dot{G}_{\gamma_m} \tag{5.9}$$

Suppose now that p' extends p and identifies $\dot{r}$ to be some $r \in P_\gamma$. Then $p' \Vdash (\check{r} \restriction \gamma_m) \in \dot{G}_{\gamma_m}$ for every $m \in \omega$, so since the forcing notions are separative, $p' \leq (r \restriction \gamma_m)$ for every m. We also have $p' \Vdash \check{r} \in M$ and thus since the domain of r is countable, it is a subset of M. In particular $\operatorname{dom}(r) \subseteq \gamma \cap M$. This fact together with (5.9), gives us that $p' \leq r$, so $p' \Vdash r \in \dot{G}_\gamma$. Since this is true for every p' extending p that identifies $\dot{r}$, we have found an (M, P_γ)-generic condition p such that $p \restriction \gamma_0 = p_{\gamma_0}$ and $p \Vdash \dot{r} \in \dot{G}_\gamma$. □

Theorem 5.3.2. *If $\langle P_\alpha, \dot{Q}_\alpha : \alpha \leq \gamma\rangle$ is a countable support iteration of proper forcing notions, then P_γ is a proper forcing.*

Proof. let λ be sufficiently large, $M \prec H_\lambda$ be countable such that $P_\gamma, \gamma, \langle P_\alpha, \dot{Q}_\alpha : \alpha \leq \gamma\rangle \in M$, and let $r \in M \cap P_\gamma$. Then let $\gamma_0 = 0$, let $\dot{r}$ be the canonical name for r over the forcing $P_0 = \{\emptyset\}$, and let $p_0 = \emptyset$. Notice that p_0 is (M, P_0)-generic, and $p_0 \Vdash_{P_0}$ “$\dot{r} \restriction 0 \in \dot{G}_0$ and $\dot{r} \in \check{M} \cap \check{P}_\gamma$. Then by the previous lemma, there is some (M, P_γ)-generic condition p such that $p \Vdash_{P_\gamma} \dot{r} \in \dot{G}_\gamma$. Since p_γ is separative it is clear now that $p \leq r$, so P_γ is a proper forcing. □

5.4 Dominating Reals in Iterated Proper Forcing

We get some other really nice properties from the countable support iteration of proper forcing notions. The first important one happens at limit steps of uncountable cofinalities. This lemma was adapted to match our style from lemma 1.5.7 in [3].

Lemma 5.4.1. *Let $\langle P_\alpha, \dot{Q}_\alpha : \alpha \leq \gamma\rangle$ be a countable support iteration of proper forcing notions, and let γ be a limit ordinal of uncountable cofinality, then*

$$1_{P_\gamma} \Vdash_{P_\gamma} \omega^\omega \cap V[\dot{G}_\gamma] = \bigcup_{\alpha<\gamma} (\omega^\omega \cap V[\dot{G}_\alpha])$$

(i.e. no new reals are added at the limit steps of uncountable cofinality.)

Proof. Let $\dot{f} \in V$ be a P_γ name for a function in ω^ω. By **lemma 2.4.4** we can assume $\dot{f}$ is a 'nice' name in the sense that if $x \in \dot{f}$ then x has the form $\langle \check{a}, p\rangle$ for some $a \in \omega\times\omega$, and some $p \in P_\gamma$. Now for each $m \in \omega$ define $D_m := \{p \in P_\gamma :$ "p decides $\dot{f}(\check{m})$"$\}$, and for γ sufficiently large, let $M \prec H_\lambda$ be a countable submodel with $\{D_m : m \in \omega\}$, $\dot{f}$, P_γ, etc $\in M$. Since M is countable, and the cofinality of γ is uncountable, $\alpha := \gamma \cap M$ is strictly less than γ, and $\hat{f} := \dot{f} \cap M$ is a P_α-name. Now let $p \in P_\gamma$, and by the properness of P_γ, let $q \leq p$ be a (M, P_γ)-generic condition. We claim that $q \Vdash \hat{f} = \dot{f}$. It is clear that $q \Vdash \hat{f} \subseteq \dot{f}$, because as names $\hat{f} \subseteq \dot{f}$. So let G be a generic filter on P_γ such that $q \in G$, and suppose $\langle m, n\rangle \in \dot{f}^G$. We know by the genericity of q that $D_m \cap M$ is predense below q, and thus there is some $q' \in D_m \cap M \cap G$. Then since q' decides $\dot{f}(\check{m})$, and $\dot{f}$ is a nice name, we know $\langle \langle m, n\rangle\check{}, q'\rangle \in \dot{f}$. Since $q' \in M$, $\langle \langle m, n\rangle\check{}, q'\rangle \in \hat{f}$, and therefore $\langle m, n\rangle \in \hat{f}^G$. □

It would be nice if we could get that property for all limit steps, but unfortunatly we needed the uncountable cofinality to restrict the domain of the name of the new real. We are able to say something about the reals that we add at limits of countable cofinality, providing we have enough information about the steps below it. We will adapt theorem 6.1.18 from [3] to our style.

Lemma 5.4.2. *Let $\langle P_\alpha, \dot{Q}_\alpha : \alpha < \gamma\rangle$ be a countable support iteration of proper forcing notions, and let γ be a limit ordinal. Suppose that P_α does not add a dominating real for all $\alpha < \gamma$. Then P_γ does not add a dominating real.*

Proof. If γ has uncountable cofinality, then by the previous lemma, no new real is added in the γ step of the iteration, so in particular, no new dominating real is added. Suppose now instead that the cofinality of γ is $\aleph_0$, and let $\langle \gamma_n : n \in \omega\rangle$ be a cofinal sequence of ordinals in γ with $\gamma_0 = 0$. Let $\dot{g}$ be a P_γ name for a real, and for each $m \in \omega$ let $D_m := \{p \in P_\gamma :$ "p decides $\dot{g} \restriction \check{m}$"$\}$. Now let λ be sufficiently large, and let $M \prec H_\lambda$ be a countable submodel with $\{D_m : m \in \omega\}$, $\{\gamma_n : n \in \omega\}$, $\langle P_\alpha, \dot{Q}_\alpha : \alpha < \gamma\rangle$, P_γ, $\dot{g} \in M$. Now let $\langle f_i : i \in \omega\rangle$ be an enumeration of $\omega^\omega \cap M$, and define a function $f \in \omega^\omega$ by $f(n) := \max\{f_i(n) + 1 : i \leq n\}$. Let $\dot{p}_0 \in P_\gamma$ be the canonical P_0-name for some forcing condition $p_0 \in P_\gamma$, and let $q_0 = \emptyset$. By induction we will find a $q \leq p_0$ such that $q \Vdash \check{f} \not\leq^* \dot{g}$.

Suppose we have already defined a P_{γ_n}-name $\dot{p}_n$ and $q_n \in P_{\gamma_n}$ such that there is some P_{γ_n}-name $\dot{m}$ for a natural number such that $q_n \Vdash_{P_{\gamma_n}} \check{n} \leq \dot{m}$, (for $n = 0$, let $\dot{m} = 0$) and

$$\begin{aligned} q_n \Vdash \text{“}&\dot{p}_n \in \check{P}_\gamma \cap \check{M} \\ &\dot{p}_n \restriction \gamma_n \in \dot{G}_n \\ &\dot{p}_n \in D_{\dot{m}} \\ &\dot{p}_n \leq \dot{p}_{n-1} \text{ (for } n > 0)\text{”} \end{aligned}$$

Then define $p_{n,0} := P_{\gamma_n}$, and for each $m \in \omega$, choose a P_{γ_n}-name (just like in the properness extension lemma) such that for each m

$$\begin{aligned} q_n \Vdash \text{“}&\dot{p}_{n,m} \in \check{P}_\gamma \cap \check{M} \\ &\dot{p}_{n,m} \restriction \gamma_n \in \dot{G}_n \\ &\dot{p}_{n,m+1} \leq \dot{p}_{n,m} \\ &\dot{p}_{n,m} \in D_m\text{”} \end{aligned}$$

Then let G_n be an arbitrary generic filter on P_{γ_n} such that $q_n \in G_n$, and in $M[G_n]$ define $g_n \in \omega^\omega$ by $g_n(m) := l$ if and only if $\dot{p}_{n,m+1}^{G_n} \Vdash \dot{g}(\check{m}) = \check{l}$, and since $g_n \in M[G_n]$ we can choose a P_{γ_n}-name $\dot{g}_n \in M$ such that for each m, $q_n \Vdash_{P_{\gamma_n}} (\dot{p}_{m,n} \Vdash_{P_\gamma} \dot{g}_n(\check{m}) = \dot{g}(\check{m}))$. Then since $\dot{g}_n$ is a P_{γ_n} name for a real, by the assumption of the lemma, it is not a dominating real. So specifically $H_\lambda \models \exists \dot{h} \in \omega^\omega \ (1_{P_{\gamma_n}} \Vdash_{P_{\gamma_n}} \dot{h} \not\leq^* \dot{g}_n)$ where $\dot{h}$ is the P_{γ_n}-name for a real in $V[G_n]$. Since M is an elementary submodel of H_λ, we can find such an $\dot{h}$ in M. Since $\dot{h}$ is a name for a real, and $q \Vdash_{P_{\gamma_n}} \dot{h} \in \check{M}$, there is some P_{γ_n}-name $\dot{n}$ such that $q_n \Vdash_{P_{\gamma_n}} \dot{h} = f_{\dot{n}}$ (from our enumeration of the reals in M). The result is that $q_n \Vdash_{P_{\gamma_n}} \forall m \geq \dot{n} \ (\check{f}(m) > \dot{h}(m))$. In particular, there is a P_{γ_n}-name $\dot{m}$ for a natural number such that

$$\begin{aligned} q_n \Vdash_{P_{\gamma_n}} \text{“}&\check{n}, \dot{n} \leq \dot{m} \\ &\dot{g}_n(\dot{m}) < \dot{h}(\dot{m}) < \check{f}(\dot{m})\text{”} \end{aligned}$$

So in particular:

$$q_n \Vdash_{P_{\gamma_n}} \text{“}\check{f} \not\leq^*_{\check{n}} \dot{g}_n\text{”}$$

□

Now define $\dot{p}_{n+1} := \dot{p}_{n,\dot{m}}$, and by the properness extension lemma, choose a $(M, P_{\gamma_{n+1}})$-generic q_{n+1} such that $q_{n+1} \restriction n = q_n$, and

$$\begin{aligned} q_{n+1} \Vdash_{P_{\gamma_n}} \text{“}&\dot{p}_{n+1} \in \check{P}_\gamma \cap \check{M} \\ &\dot{p}_{n+1} \restriction \gamma_{n+1} \in \dot{G}_{n+1} \\ &\dot{p}_{n+1} \in D_{\dot{m}} \\ &\dot{p}_{n+1} \leq \dot{p}_n\text{”} \end{aligned}$$

Then as before we define $q := \bigcup_{n \in \omega} q_n$ and claim that $q \Vdash \check{f} \not\leq^* \dot{g}$. Just as in the properness extension lemma, we get that for every n, $q \Vdash_{P_\gamma} \dot{p}_n \in \dot{G}_\gamma$. Along with the fact that for every n, there is a P_{γ_n} name $\dot{m}$ for a natural number greater than n such that $q \Vdash (\dot{p}_n \Vdash \dot{g}_n(\dot{m}) = \dot{g}(\dot{m}))$, and $q \Vdash \dot{g}_n(\dot{m}) < \dot{h}(\dot{m}) < \check{f}(\dot{m})$, we have that for each $n \in \omega$, $q \Vdash \check{f} \not\leq^*_{\check{n}} \dot{g}$. Thus $q \Vdash \check{f} \not\leq^* \dot{g}$, the desired result.

Finally, we would like to point out one last result that we will not prove here.

Theorem 5.4.3. *Let $\langle P_\alpha, \dot{Q}_\alpha : \alpha \leq \gamma \rangle$ be a countable support iteration of length $\gamma \leq \omega_2$ of proper forcings of size at most $\aleph_1$. Then P_γ satisfies the $\aleph_2$-chain condition.*

Proof. See section 2.2 in [1]. □

5.5 An Application of Iterated Proper Forcing

With all of these tools built up, it would be a shame not to give an application. We demonstrate here model 7.5.1 from [3]. We will construct this model from forcing notions we introduced in the Generic Reals chapter, and will compute enough values on the corresponding Cichoń diagram, to know all 10 of them (with the help of the inequalities mentioned in the introduction of the diagram).

Example 5.5.1. *Let $\langle P_\alpha, \dot{Q}_\alpha : \alpha < \omega_2\rangle$ be the countable support iteration where:*

1. *If α is even, $\dot{Q}_\alpha = \dot{\mathbb{B}}_C$, a P_α-name for the Solovay forcing.*
2. *If α is odd, $\dot{Q}_\alpha = \dot{\mathbb{C}}_2$, a P_α-name for the Cohen forcing.*

From the previous lemma, this model has the $\aleph_2$-chain condition, so cardinals greater than and equal to $\aleph_2$ are preserved. We also know that $\aleph_1$ is preserved because the is a proper forcing. For the following lemmas, let G_{ω_2} be a P_{ω_2}-generic filter over V.

Lemma 5.5.2. *In $V[G_{\omega_2}]$ we have $cov(\mathcal{M}) = cov(\mathcal{N}) = \aleph_2$.*

Proof. Let $\langle A_\gamma : \gamma < \lambda\rangle$ where $\lambda < \omega_2$ be a sequence of meager subsets of $\mathbb{R}$ in $V[G_{\omega_2}]$. Now since each A_γ is meager there is a G_δ set, B_γ, that can be expressed as the intersection of countably many dense open sets such that $A_\gamma \cap B_\gamma = \emptyset$. Instead of being the actual set, let B_γ be a G_δ-code associated with the given G_δ set. Since G_δ-codes are countable collections of countable collections of pairs of rational numbers, we can see that the set of all G_δ-codes in V is in bijection with the subsets of ω in V. Since no new reals, and in particular no new subsets of ω are added in the limit step P_{ω_2}, we have for each $\gamma < \lambda$, an $\alpha_\gamma < \omega_2$ such that $B_\gamma \in V[G_{\alpha_\gamma}]$. Define $\alpha := \sup\{\alpha_\gamma : \gamma < \lambda\}$, and notice that since λ and each of the α_γ's are less than ω_2, and ω_2 is regular, $\alpha < \omega_2$. This means all of these G_δ-codes occur already in $V[G_\alpha]$. Now let $\beta = \begin{cases} \alpha & \text{if } \alpha \text{ is odd} \\ \alpha+1 & \text{if } \alpha \text{ is even} \end{cases}$, then $f_\beta := \bigcup_{p\in G_{\beta+1}} p(\beta)^{G_\beta}$ is a Cohen real generic over $V[G_\beta]$, and is thus an element of each set $B_\gamma^{V[G_{\beta+1}]} \subseteq B_\gamma^{V[G_{\omega_2}]}$. In particular $f_\beta \notin \bigcup_{\gamma<\lambda} A_\gamma$, so $\langle A_\gamma : \gamma < \lambda\rangle$ can not be a cover of $\mathbb{R}^{V[G_{\omega_2}]}$, and thus $\mathrm{cov}(\mathcal{M}) \geq \aleph_2$.

To show that $\mathrm{cov}(\mathcal{N}) = \aleph_2$, we start with a sequence of measure zero sets $\langle A_\gamma : \gamma < \lambda\rangle$ where $\lambda < \omega_2$, and for each choose a G_δ-code $B_\gamma \in V[G_{\omega_2}]$ for a null set in $V[G_{\omega_2}$ such that $A_\gamma \subseteq B_\gamma^{V[G_{\omega_2}]}$. Choose as above an $\alpha < \omega_2$ such that all of these G_δ codes occur in $V[G_\alpha]$. This time let $\beta = \begin{cases} \alpha+1 & \text{if } \alpha \text{ is odd} \\ \alpha & \text{if } \alpha \text{ is even} \end{cases}$, and

$$a_\beta := \bigcap_{q\in G_{\beta+1}} \left\{p^{V[G_{\beta+1}]} : p \in V[G_\beta] \text{ is an } F\text{-code} \wedge q(\beta)^{G_\beta} = p^{V[G_\beta]}\right\}$$

Then a_β is a random real generic over $V[G_\beta]$, and thus is not an element of any $B_\gamma^{V[G_{\beta+1}]}$, so is also not an element of any $B_\gamma^{V[G_{\omega_2}]}$. In particular $r_\beta \notin \bigcup_{\gamma<\lambda} A_\gamma$, so $\langle A_\gamma : \gamma < \lambda\rangle$ can not be a cover of $\mathbb{R}^{V[G_{\omega_2}]}$, and thus $\mathrm{cov}(\mathcal{N}) \geq \aleph_2$. $\square$

Now we would like to show that in this model, the bounding number is still $\aleph_1$. To do this, we need first a basic lemma about bounding numbers and forcing notions.

Lemma 5.5.3. *Suppose $P \in V$ is a forcing of size less than $\mathfrak{b}$. Then for every function $f \in \omega^\omega \cap V[G]$, there exists a function $g_f \in V$ such that if $h \in V$ and $h \leq^* f$ then $h \leq^* g_f$*

Proof. Let $\dot{f} \in V$ be a P-name for f, then for each $p \in P$ define the function f_p by

$$f_p(n) := \min\{k : \exists q \leq p\ (q \Vdash_P \dot{f}(\check{n}) = \check{k})\}$$

Now since $\{f_p : p \in P\}$ has cardinality less than $\mathfrak{b}$, we can find a function g_f that dominates this family. Notice now that if $p \Vdash_P h \leq^* \dot{f}$, then $h \leq^* f_p \leq^* g_f$. $\square$

Lemma 5.5.4. *If CH holds in V, then in $V[G_{\omega_2}]$ we have $\mathfrak{b} = \aleph_1$.*

Proof. Since we have assumed CH in V, $\text{card}^V(\omega^\omega \cap V) = \aleph_1$, so we will show that $\omega^\omega \cap V$ is unbounded in $V[G_{\omega_2}]$ by induction, and this is enough to show $\mathfrak{b} = \aleph_1$. First, for the successor steps, assume $\omega^\omega \cap V$ is unbounded in $V[G_\alpha]$.

1. If α is even:

   $V[G_{\alpha+1}]$ is a random extension of $V[G_\alpha]$, so given any $f \in \omega^\omega \cap V[G_{\alpha+1}]$, there is some $g \in V[G_\alpha]$ that dominates f. By induction $\omega^\omega \cap V$ is unbounded in $V[G_\alpha]$, so there is some $h \in V$ that is not bounded by g, and thus is not bounded by f.

2. If α is odd:

   $V[G_{\alpha+1}]$ is a Cohen extension of $V[G_\alpha]$, which is generated by the countable Cohen forcing notion. In particular it is generated by a forcing notion whose cardinality is less than $\mathfrak{b}$, so given any $f \in \omega^\omega \cap V[G_{\alpha+1}]$, there is some $g \in V[G_\alpha]$ such that if $h \in \omega^\omega \cap V[G_\alpha]$ and $h \leq^* f$, then $h \leq^* g$. Since we assumed that $\omega^\omega \cap V$ is unbounded in $V[G_\alpha]$, there is some $h \in V$ that is not bounded by g, and thus is not bounded by f.

Now assume that δ is a limit ordinal, and that for all $\alpha < \delta$, $\omega^\omega \cap V$ is unbounded in $V[G_\alpha]$, then by **lemma 5.4.2** no new dominating real is added at this step, so $\omega^\omega \cap V$ is still unbounded in $V[G_\delta]$.

□

This model has the following Cichoń diagram

$$\begin{tikzcd}
 & \aleph_2 \arrow[r] & \aleph_2 \arrow[r] & \aleph_2 \arrow[r] & \aleph_2 \arrow[r] & 2^{\aleph_0} = \aleph_2 \\
 & & \aleph_1 \arrow[u] \arrow[r] & \aleph_2 \arrow[u] & & \\
\aleph_1 \arrow[r] & \aleph_1 \arrow[r] \arrow[uu] & \aleph_1 \arrow[r] \arrow[u] & \aleph_2 \arrow[r] \arrow[u] & \aleph_2 \arrow[uu] &
\end{tikzcd}$$

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